SOPHISTICATED MONETARY POLICIES*

Andrew Atkeson
Varadarajan V. Chari
Patrick J. Kehoe

In standard monetary policy approaches, interest-rate rules often produce indeterminacy. A sophisticated policy approach does not. Sophisticated policies depend on the history of private actions, government policies, and exogenous events and can differ on and off the equilibrium path. They can uniquely implement any desired competitive equilibrium. When interest rates are used along the equilibrium path, implementation requires regime-switching. These results are robust to imperfect information. Our results imply that the Taylor principle is neither necessary nor sufficient for unique implementation. They also provide a direction for empirical work on monetary policy rules and determinacy.

I. Introduction

The now-classic Ramsey (1927) approach to policy analysis under commitment specifies the set of instruments available to policy makers and finds the best competitive equilibrium outcomes given those instruments. This approach has been adapted to situations with uncertainty, by Barro (1979) and Lucas and Stokey (1983), among others, by specifying the policy instruments as functions of exogenous events.[1]

Although the Ramsey approach has been useful in identifying the best outcomes, it needs to be extended before it can be used to guide policy. Such an extension must describe what would happen for every history of private agent actions, government policies, and exogenous events. It should also structure policy in such a way that policy makers can ensure that their desired outcomes occur.

Here, we provide such an extended approach. To construct it, we extend the language of Chari and Kehoe (1990) in a natural fashion by describing private agent actions and government policies as functions of the histories of those actions and policies as well as of exogenous events. The key to our approach is our

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1. The Ramsey approach has been used extensively to discuss optimal monetary policy. See, among others, the work of Chari, Christiano, and Kehoe (1996); Schmitt-Grohé and Uribe (2004); Siu (2004); and Correia, Nicolini, and Teles (2008).

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requirement that for all histories, including those in which private agents deviate from the equilibrium path, the continuation outcomes constitute a continuation competitive equilibrium.[2] We label such policy functions *sophisticated policies* and the resulting equilibrium a *sophisticated equilibrium*. If policies can be structured to ensure that the desired outcomes occur, then we say that the policies *uniquely implement* the desired outcome.

Here we describe this approach and use it to analyze an important outstanding question in monetary economics: How should policy be designed in order to avoid indeterminacy and achieve unique implementation? It has been known, at least since the work of Sargent and Wallace (1975), that when interest rates are the policy instrument, many ways of specifying policy lead to indeterminate outcomes including multiple equilibria. Indeterminacy is risky because some of those outcomes can be bad, including hyperinflation. Researchers thus agree that designing policies that achieve unique implementation is desirable. Here we demonstrate that our sophisticated policy approach does that for monetary policy.

We illustrate our approach in two standard monetary economies: a simple sticky-price model with one-period price-setting and a sticky-price model with staggered price-setting (often referred to as the *New Keynesian model*). For both, we show that, under sufficient conditions, any outcome of a competitive equilibrium can be uniquely implemented by appropriately constructed sophisticated policies. In particular, the Ramsey equilibrium can be uniquely implemented.

In the two model economies, we construct central bank policies that uniquely implement a desired competitive equilibrium in the same basic way. Along the equilibrium path, we choose the policies to be those given by the desired competitive equilibrium. We structure the policies off the equilibrium path, the *reversion* policies, to discourage deviations. Specifically, if the average choice of private agents deviates from that in the desired equilibrium, then we choose the reversion policies so that the optimal choice, or *best response,* of each individual agent is different from the average choice.

One way to see why such reversion policies can eliminate multiplicity is to recall how multiple equilibria arise in the first

2. This requirement is the natural analog of subgame perfection in an environment in which private agents are competitive. In this sense, our equilibrium concept is the obvious one for our macroeconomic environment.

place. At an intuitive level, they arise if, when each agent believes that all other agents will choose some particular action other than the desired one, each agent finds it optimal to go along with the deviation by also picking that particular action. Our construction of reversion policies breaks the self-fulfilling nature of such deviations. It does so by ensuring that even if an agent believes that all other agents are choosing a particular action that differs from the desired action, the central bank policy makes it optimal for that agent not to go along with that deviation.

When such reversion policies can be found, we say that the best responses are *controllable*. A sufficient condition for controllability is that policies can be found such that after a deviation the continuation equilibrium is unique and varies with policy. Variation with policy typically holds, so if policies can be found under which the continuation equilibrium is unique (somewhere), then we have unique implementation (everywhere). This sufficient condition suggests a simple way to state our message in a general way: uniqueness somewhere generates uniqueness everywhere.

One concern with our construction of sophisticated policies is that it apparently relies on the idea that the central bank perfectly observes private agents' actions and thus can detect any deviation. We show that this concern is unwarranted: our results are robust to imperfect information about private agents' actions. Specifically, with imperfect detection of deviations, sophisticated policies can be designed that have unique equilibria that are close to the desired outcomes when the detection error is small and that converge to the desired equilibria as the detection error goes to zero.

The approach proposed here suggests an operational guide to policy making: First use the Ramsey approach to determine the best competitive equilibrium, and then check whether in that situation, best responses are controllable. If they are, then sophisticated policies of the kind we have constructed can uniquely implement the Ramsey outcome. If best responses are not controllable, then the only option is to accept indeterminacy.

Our work here is related to previous work on the problem of indeterminacy in monetary economies (Wallace 1981; Obstfeld and Rogoff 1983; King 2000; Benhabib, Schmitt-Grohé, and Uribe 2001; Christiano and Rostagno 2001; Svensson and Woodford 2005). The previous work pursues an approach different from ours (and from that in the microeconomic literature on implementation); we call it *unsophisticated implementation.* The basic idea of that approach is to specify policies as functions of the history and check only to see whether the period-zero competitive equilibrium is unique.

Unsophisticated implementation has been criticized in the macroeconomic and the microeconomic literature. For example, in the macroeconomic literature, Kocherlakota and Phelan (1999), Bassetto (2002), Buiter (2002), and Ljungqvist and Sargent (2004) criticize this general idea in the context of the fiscal theory of the price level; Bassetto (2005) criticizes it in the context of a simple tax example; and Cochrane (2007) criticizes it in the context of the literature on monetary policy rules. In the microeconomic literature, Jackson (2001) criticizes a related approach to implementation.

In our view, unsophisticated implementation is deficient because it does not describe how the economy will behave after a deviation by private agents from the desired outcome. This deficiency leaves open the possibility that the approach achieves *implementation via nonexistence*. By this phrase, we mean an approach that specifies policy actions under which no continuation equilibrium exists after private agent deviations.

We agree with those who argue that implementation via nonexistence trivializes the implementation problem. To see why it does, consider the following policy rule: If private agents choose the desired outcome, then continue with the desired policy; if private agents deviate from the desired outcome, then forever after set government spending at a high level and taxes at zero. Clearly, under this policy rule, any deviation from the desired outcome leads to nonexistence of equilibrium, and hence, we trivially have implementation via nonexistence. We find this way of achieving implementation unpalatable.

Our approach, in contrast, insists that policies be specified such that a competitive equilibrium exists after any deviation. We achieve implementation in the traditional microeconomic sense—by discouraging deviations, not by nonexistence. In our approach, policies are specified so that even if an individual agent believes that all other agents will deviate to some specific action, that individual agent finds it optimal to choose a different action. Our approach not only ensures that the continuation equilibria always exist, but also has the desirable property that the reversion policies are not extreme in any sense. That is, after deviations, our reversion policies do not threaten the private economy with dire outcomes such as hyperinflation; they simply bring inflation back to the desired path.

Despite the shortcomings of the unsophisticated implementation approach, this literature has made two contributions that we find useful. One is the idea of *regime-switching*. This idea dates back at least to Wallace (1981) and has been used by Obstfeld and Rogoff (1983), Benhabib, Schmitt-Grohé, and Uribe (2001), and Christiano and Rostagno (2001). The basic idea in, say, Benhabib, Schmitt-Grohé, and Uribe (2001) is that if the economy embarks on an undesirable path, then the monetary and fiscal policy regime switches in such a way that the government's budget constraint is violated, and the undesirable path is not an equilibrium.

The other useful contribution of the literature on unsophisticated implementation is what Cochrane (2007) calls the *King rule*. This rule seeks to implement a desired equilibrium through an interest-rate policy that makes the difference between the interest rate and its desired equilibrium level a linear function of the difference between inflation and its desired equilibrium level, with a coefficient greater than 1. This idea dates back to at least King (2000) and has been used by Svensson and Woodford (2005). As we show here, the King rule, like other rules that use interest rates for all histories, namely, *pure interest-rate rules*, always leads to indeterminacy in our simple model and does so for a large class of parameters in our staggered price-setting model as well.

We build on these two contributions by considering a *King–money hybrid rule*: When private agents deviate from the equilibrium path, the central bank uses the King rule for small deviations and switches regimes (from interest rates to money) for large deviations. Notice that with this rule, under our definition of equilibrium, outcomes return to the desired outcome path in the period after the deviation. In this sense, our hybrid rule achieves unique implementation without threatening agents with dire outcomes.

Our work here is also related to another substantial literature that aims to find monetary policy rules which eliminate indeterminacy. (See, for example, McCallum [1981] and, more recently, Woodford [2003].) The recent literature argues that to achieve a unique outcome, interest-rate rules should follow the *Taylor principle*: interest rates relative to exogenously specified levels should rise more than one for one when inflation rates rise relative to their exogenously specified levels.

We show here that adherence to the Taylor principle is neither necessary nor sufficient for unique implementation. It is not necessary because the sophisticated policy approach can uniquely implement any desired competitive equilibrium outcome, including outcomes in which, along the equilibrium path, the central bank follows an interest-rate rule that violates the Taylor principle. It is not sufficient because pure interest-rate rules may lead to indeterminacy even if they satisfy the Taylor principle.

Notwithstanding these considerations, our analysis of the King–money hybrid rule does lend support to the idea that adherence to the Taylor principle can sometimes help achieve unique implementation. Specifically, this is true within the class of King–money hybrid rules when the Taylor principle is used in the region where the King part of the rules applies.

Our findings also cast light on empirical investigations of determinacy based on the Taylor principle. We argue that, under the set of assumptions made explicit in the literature, inferences about determinacy based on existing estimation procedures should be treated skeptically. For our simple model economies, we provide assumptions under which such inferences can be confidently made. Although there is some hope that such inference may be possible in more interesting applied examples using variants of our assumptions, difficult challenges remain.

Using sophisticated policies is our proposed way to eliminate indeterminacy when setting monetary policy. For some other recent proposals, see the work of Bassetto (2002) and Adão, Correia, and Teles (2007).

II. A Simple Model with One-Period Price-Setting

We begin by illustrating the basic idea of our construction of sophisticated policies using a simple model with one-period price-setting. The dynamical system associated with the competitive equilibrium of this model is straightforward, which lets us focus on the strategic aspects of sophisticated policies. With this model, we demonstrate that any desired outcome of a competitive equilibrium can be uniquely implemented by sophisticated policies with reversion to a money regime. We show that pure interest-rate rules, which exclusively use interest rates as the policy instrument, cannot achieve unique implementation. Finally, we show that reversion to a particular hybrid rule, which uses interest rates as the policy instrument for small deviations and money for large deviations, can achieve unique implementation.

The model we analyze here is a modified version of the basic sticky-price model with a New Classical Phillips curve (as in

Woodford [2003, Chap. 3, Sect. 1.3]). In order to make our results comparable to those in the literature, we here describe a simple, linearized version of the model. In Atkeson, Chari, and Kehoe (2009), we describe the general equilibrium version that, when linearized, produces the equilibrium conditions studied here.

II.A. The Determinants of Output and Inflation

Consider a monetary economy populated by a large number of identical, infinitely lived consumers, a continuum of producers, and a central bank. Each producer uses labor to produce a differentiated good on the unit interval. A fraction of producers $j \in [0, \alpha)$ are *flexible-price* producers, and a fraction $j \in [\alpha, 1]$ are *sticky-price* producers.

In this economy, the timing within a period t is as follows. At the beginning of the period, sticky-price producers set their prices, after which the central bank chooses its monetary policy by setting one of its instruments, either interest rates or the quantity of money. Two shocks, η_t and ν_t, are then realized. We interpret the shock η_t as a *flight to quality* shock that affects the attractiveness of government debt relative to private claims and the shock ν_t as a *velocity* shock. At the end of the period, flexible-price producers set their prices, and consumers make their decisions.

Now we develop necessary conditions for a competitive equilibrium in this economy and then, in the next section, formally define a competitive equilibrium. Here and throughout, we express all variables in log-deviation form. This way of expressing variables implies that none of our equations will have constant terms.

Consumer behavior in this model is summarized by an intertemporal Euler equation and a cash-in-advance constraint. We can write the linearized Euler equation as

$$y_t = E_t\,[y_{t+1}] - \psi\,(i_t - E_t\,[\pi_{t+1}]) + \eta_t, \tag{1}$$

where y_t is aggregate output, i_t is the nominal interest rate, η_t (the flight to quality shock) is an i.i.d. mean-zero shock with variance var(η), and $\pi_{t+1} = p_{t+1} - p_t$ is the inflation rate from time period t to $t+1$, where p_t is the aggregate price level. The parameter ψ determines the intertemporal elasticity, and E_t denotes the expectations of a representative consumer given that consumer's information in period t, which includes the shock η_t.

The cash-in-advance constraint, when first-differenced, implies that the relationships among inflation π_t, money growth μ_t, and output growth $y_t - y_{t-1}$ are given by a quantity equation of the form

$$\pi_t = \mu_t - (y_t - y_{t-1}) + \nu_t, \tag{2}$$

where ν_t (the velocity shock) is an i.i.d. mean-zero shock with variance var(ν).

We turn now to producer behavior. The optimal price set by an individual flexible-price producer j satisfies

$$p_{ft}(j) = p_t + \gamma y_t, \tag{3}$$

where the parameter γ is the elasticity of the equilibrium real wage with respect to output (often referred to in the literature as *Taylor's* γ). The optimal price set by a sticky-price producer j satisfies

$$p_{st}(j) = E_{t-1}\left[p_t + \gamma y_t\right], \tag{4}$$

where E_{t-1} denotes expectations at the beginning of period t before the shocks η_t and ν_t are realized. The aggregate price level p_t is a linear combination of the prices p_{ft} set by the flexible-price producers and the prices p_{st} set by the sticky-price producers and is given by

$$p_t = \int_0^{\alpha} p_{ft}(j)\,dj + \int_{\alpha}^{1} p_{st}(j)\,dj. \tag{5}$$

Using language from game theory, we can think of equations (3) and (4) as akin to the best responses of the flexible- and sticky-price producers given their beliefs about the aggregate price level and aggregate output.

In this model, the flexible-price producers are strategically uninteresting. Their expectations about the future have no influence on their decisions; their prices are set mechanically according to the static considerations reflected in (3). Thus, in all that follows, equation (3) will hold on and off the equilibrium path, and we can think of $p_{ft}(j)$ as being residually determined by (3) and substitute out for $p_{ft}(j)$. To do so, substitute (3) into (5) and solve for p_t to get

$$p_t = \kappa y_t + \frac{1}{1-\alpha}\int_{\alpha}^{1} p_{st}(j)\,dj, \tag{6}$$

where $\kappa = \alpha\gamma/(1-\alpha)$.

We follow the literature and express the sticky-price producers' decisions in terms of inflation rates rather than price levels. To do so, let $x_t(j) = p_{st}(j) - p_{t-1}$, and rewrite (4) as

$$x_t(j) = E_{t-1}\left[\pi_t + \gamma y_t\right]. \tag{7}$$

For convenience, we define

$$x_t = \frac{1}{1-\alpha}\int_{\alpha}^{1} x_t(j)\, dj \tag{8}$$

to be the average price set by the sticky-price producers relative to the aggregate price level in period $t-1$, so that we can rewrite (7) as

$$x_t = E_{t-1}\left[\pi_t + \gamma y_t\right]. \tag{9}$$

We can also rewrite (6) as

$$\pi_t = \kappa y_t + x_t. \tag{10}$$

Consider now the setting of monetary policy in this model. When the central bank sets its policy, it has to choose to operate under either a *money regime* or an *interest-rate regime*. In the money regime, the central bank's policy instrument is money growth μ_t; it sets μ_t, and the nominal interest rate i_t is residually determined from the Euler equation (1) after the realization of the shock η_t. In the interest-rate regime, the central bank's instrument is the interest rate; it sets i_t, and money growth μ_t is residually determined from the cash-in-advance constraint (2) after the realization of the shock ν_t. Of course, in both regimes, the Euler equation and the cash-in-advance constraint both hold.

II.B. Competitive Equilibrium

Now we define a notion of *competitive equilibrium* for the simple model in the spirit of the work of Barro (1979) and Lucas and Stokey (1983). In this equilibrium, allocations, prices, and policies are all defined as functions of the history of exogenous events, or *shocks*, $s^t = (s_0, \ldots, s_t)$, where $s_t = (\eta_t, \nu_t)$.

Sticky-price producer decisions and aggregate inflation and output levels can be summarized by $\{x_t(s^{t-1}), \pi_t(s^t), y_t(s^t)\}$. In terms of the policies, we let the regime choice and the policy choice within the regime be $\delta_t(s^{t-1}) = (\delta_{1t}(s^{t-1}), \delta_{2t}(s^{t-1}))$, where the first argument $\delta_{1t}(s^{t-1}) \in \{M, I\}$ denotes the regime choice, either money (M) or the interest rate (I), and the second argument denotes the policy choice within the regime, either money growth $\mu_t(s^{t-1})$ or the interest rate $i_t(s^{t-1})$. If the money regime is chosen in t, then the interest rate is determined residually at the end of that period, whereas if the interest-rate regime is chosen in t, then the money growth rate is determined residually at the end of the period. Let $\{a_t(s^t)\} = \{x_t(s^{t-1}), \delta_t(s^{t-1}), \pi_t(s^t), y_t(s^t)\}$ denote a collection of allocations, prices, and policies in this competitive equilibrium.

Such a collection is a *competitive equilibrium* given y_{-1} if it satisfies (i) consumer optimality, namely, (1) and (2) for all s^t; (ii) optimality by sticky-price producers, namely, (9) for all s^{t-1}; and (iii) optimality by flexible-price producers, namely, (10) for all s^t.

We also define a *continuation competitive equilibrium* starting from any point in time. For example, consider the beginning of period t with state variables s^{t-1} and y_{t-1}. A collection of allocations, prices, and policies

$$\begin{aligned}\{a(s^{t-1}, y_{t-1})\}_{r\geq t} = \{&x_r(s^{r-1} \mid s^{t-1}, y_{t-1}), \delta_r(s^{r-1} \mid s^{t-1}, y_{t-1}),\\ &\pi_r(s^r \mid s^{t-1}, y_{t-1}), y_r(s^r \mid s^{t-1}, y_{t-1})\}_{r\geq t}\end{aligned}$$

is a continuation competitive equilibrium from (s^{t-1}, y_{t-1}) if it satisfies the three conditions of a competitive equilibrium above for all periods starting from (s^{t-1}, y_{t-1}). In this definition, we effectively drop the equilibrium conditions from period 0 through period $t-1$. This notion of a continuation competitive equilibrium from the beginning of period t onward is very similar to that of a competitive equilibrium from the beginning of period 0 onward, except that the initial conditions are now given by (s^{t-1}, y_{t-1}).

We define a continuation competitive equilibrium that starts at the end of period t from $(s^{t-1}, y_{t-1}, x_t, \delta_t, s_t)$ in a similar way. This latter definition requires optimality by consumers and flexible-price producers from s^t onward and optimality by sticky-price producers from s^{t+1} onward. Note that this equilibrium must satisfy all the conditions of a continuation competitive equilibrium that starts at the beginning of period t, except for the sticky-price optimality condition in period t, namely, (9) in period t.

Finally, a continuation competitive equilibrium starting at the beginning of period 0 is simply a competitive equilibrium.

The following lemma proves that any competitive equilibrium gives rise to a New Classical Phillips curve along with some other useful properties of such an equilibrium.

LEMMA 1 (New Classical Phillips Curve and Other Useful Properties). Any competitive equilibrium must satisfy

$$\pi_t(s^t) = \kappa y_t(s^t) + E[\pi_t(s^t) \mid s^{t-1}], \tag{11}$$

which is often referred to as the *New Classical Phillips curve*;

$$E[y_t(s^t) \mid s^{t-1}] = 0 \text{ and } x_t(s^{t-1}) = E[\pi_t(s^t) \mid s^{t-1}]; \text{ and} \tag{12}$$

$$E[x_{t+1}(s^t) \mid s^{t-1}] = E[\pi_{t+1}(s^{t+1}) \mid s^{t-1}] = i_t, \tag{13}$$

where $i_t = i_t(s^{t-1})$ if the central bank uses an interest-rate regime in period t and $i_t = i_t(s^t)$ if the central bank uses a money regime in period t.

Proof. To see that $E[y_t(s^t) \mid s^{t-1}] = 0$, take expectations of (10) as of s^{t-1} and substitute into (9). Using this result in (10), we obtain $x_t(s^{t-1}) = E\left[\pi_t(s^t) \mid s^{t-1}\right]$. Substituting this result into (10) yields (11). To show (13), take expectations of the Euler equation (1) with respect to s^{t-1} and use $E[y_t(s^t) \mid s^{t-1}] = 0$ along with the law of iterated expectations to get (13). QED

A similar argument establishes that (11)–(13) hold for any continuation competitive equilibrium.

II.C. Sophisticated Equilibrium

We now turn to what we call *sophisticated equilibrium*. The definition of this concept is very similar to that for competitive equilibrium, except that here we allow allocations, prices, and policies to be functions of more than just the history of exogenous events; they are also functions of the history of both aggregate private actions and central bank policies. For sophisticated equilibrium, we require as well that for every history, the continuation of allocations, prices, and policies from that history onward constitutes a continuation competitive equilibrium.

Setup and Definition. Before turning to our formal definition, we note that our definition of sophisticated equilibrium simply specifies policy rules that the central bank must follow; it does not require that the policy rules be optimal. We specify sophisticated policies in this way in order to show that our unique implementation result does not depend on the objectives of the central bank. We think of sophisticated policies as being specified at the beginning of period 0 and of the central bank as being committed to following them.

We turn now to defining the histories that private agents and the central bank confront when they make their decisions. The public events that occur in a period are, in chronological order, $q_t = (x_t; \delta_t; s_t; y_t, \pi_t)$. Letting h_t denote the history of these events from period -1 up to and including period t, we have that $h_t = (h_{t-1}, q_t)$ for $t \geq 0$. The history $h_{-1} = y_{-1}$ is given. For notational convenience, we focus on perfect public equilibria in which the central bank's *strategy* (choice of regime and policy) is a function only of the public history.

The public history faced by the sticky-price producers at the beginning of period t when they set their prices is h_{t-1}. A strategy for the sticky-price producers is a sequence of rules $\sigma_x = \{x_t(h_{t-1})\}$ for choosing prices for every possible public history.

The public history faced by the central bank when it chooses its regime and sets either its money-growth or interest-rate policy is $h_{gt} = (h_{t-1}, x_t)$. A strategy for the central bank $\{\delta_t(h_{gt})\}$ is a sequence of rules for choosing the regime as well as the policy within the regime, either $\mu_t(h_{gt})$ or $i_t(h_{gt})$. Let σ_g denote that strategy.

At the end of period t, then, output and inflation are determined as functions of the relevant history h_{yt} according to the rules $y_t(h_{yt})$ and $\pi_t(h_{yt})$. We let $\sigma_y = \{y_t(h_{yt})\}$ and $\sigma_\pi = \{\pi_t(h_{yt})\}$ denote the sequence of output and inflation rules.

Notice that for any history, the strategies σ induce continuation outcomes in the natural way. For example, starting at some history h_{t-1}, these strategies recursively induce outcomes $\{a_r(s^r \mid h_{t-1}; \sigma)\}$. We illustrate this recursion for period t. The sticky-price producer's decision in t is given by $x_t(j, s^{t-1} \mid h_{t-1}; \sigma) = x_t(h_{t-1})$, where $x_t(h_{t-1})$ is obtained from σ_x. The central bank's decision in t is given by $\delta_t(s^{t-1} \mid h_{t-1}; \sigma) = \delta_t(h_{gt})$, where $h_{gt} = (h_{t-1}, x_t(h_{t-1}))$ and $\delta_t(h_{gt})$ is obtained from σ_g. The consumer and flexible-price producer decisions in t are given by $y_t(s^t \mid h_{t-1}; \sigma) = y_t(h_{yt})$ and $\pi_t(s^t \mid h_{t-1}; \sigma) = \pi_t(h_{yt})$, where $h_{yt} = (h_{t-1}, x_t(h_{t-1}), \delta_t(h_{t-1}, x_t(h_{t-1})))$ and $y_t(h_{yt})$ and $\pi_t(h_{yt})$ are obtained from σ_y and σ_π. Continuing in a similar way, we can recursively define continuation outcomes for subsequent periods. We can likewise define continuation outcomes $\{a_r(s^r \mid h_{gt}; \sigma)\}$ and $\{a_r(s^r \mid h_{yt}; \sigma)\}$ following histories h_{gt} and h_{yt}, respectively.

We now use these strategies and continuation outcomes to formally define our notion of equilibrium. A *sophisticated equilibrium* given the policies here is a collection of strategies (σ_x, σ_g) and allocation rules (σ_y, σ_π) such that (i) given any history h_{t-1}, the continuation outcomes $\{a_r(s^r \mid h_{t-1}; \sigma)\}$ induced by σ constitute a continuation competitive equilibrium and (ii) given any history h_{yt}, so do the continuation outcomes $\{a_r(s^r \mid h_{yt}; \sigma)\}$.[3]

Associated with each sophisticated equilibrium $\sigma = (\sigma_g, \sigma_x, \sigma_y, \sigma_\pi)$ are the particular stochastic processes for outcomes that occur along the equilibrium path, which we call *sophisticated outcomes*. These outcomes are competitive equilibrium outcomes.

We will say a policy σ_g^* *uniquely implements* a desired competitive equilibrium $\{a_t^*(s^t)\}$ if the sophisticated outcome associated with any sophisticated equilibrium of the form $(\sigma_g^*, \sigma_x, \sigma_y, \sigma_\pi)$ coincides with the desired competitive equilibrium.

A central feature of our definition of sophisticated equilibrium is our requirement that for all histories, including deviation histories, the continuation outcomes constitute a continuation competitive equilibrium. We think of this requirement as analogous to the requirement that in a subgame perfect equilibrium, the continuation strategies constitute a Nash equilibrium.

This requirement constitutes the most important difference between our approach to determinacy and that in the macroeconomic literature. Technically, one way of casting that literature's approach into our language of strategies and allocation rules is to consider the following notion of equilibrium. An *unsophisticated equilibrium* is a strategy for the central bank σ_g and allocations, policies, and prices

$$\{a_t(s^t)\} = \{x_t(s^{t-1}), \delta_t(s^{t-1}), \pi_t(s^t), y_t(s^t)\}$$

such that $\{a_t(s^t)\}$ is a period-zero competitive equilibrium and the policies induced by σ_g from $\{a_t(s^t)\}$ coincide with $\{\delta_t(s^{t-1})\}$.

In our view, unsophisticated equilibrium is a deficient guide to policy. Although an unsophisticated equilibrium does tell policy makers what to do for every history, it does not specify what will happen under their policies for every history, in particular for deviation histories. Achieving implementation using the notion of unsophisticated equilibrium is, in general, trivial. As we explained earlier, one way of achieving implementation is via nonexistence: simply specify policies so that no competitive equilibrium exists after deviation histories. We find this way of achieving implementation uninteresting.

3. In general, a sophisticated equilibrium would require that for every history (including histories in which the government acts, h_{gt}), the continuation outcomes from that history onward constitute a competitive equilibrium. Here, that requirement would be redundant because the conditions for a competitive equilibrium for h_{gt} are the same as those for h_{yt}.

Finally, to help avoid a common confusion, we stress that our definition does not require that, when there is a deviation in period t, the entire sequence starting from period 0, including the deviation in period t, constitute a period-zero competitive equilibrium. Indeed, if we achieve unique implementation, then such a sequence will not constitute a period-zero equilibrium.

Implementation with Sophisticated Policies. We focus on implementing competitive equilibria with sophisticated policies in which the central bank uses interest rates along the equilibrium path. This focus is motivated in part by the observation that most central banks seem to use interest rates as their policy instruments. Another motivation is that if the variance of the velocity shock ν_t is large, then all of the outcomes under the money regime are undesirable.

To set up our construction of sophisticated policies, recall that in our economy the only strategically interesting agents are the sticky-price producers. Their choices must satisfy a key property, that

$$x_t(h_{t-1}) = E[\pi_t(h_{yt}) + \gamma y_t(h_{yt}) \mid h_{t-1}], \tag{14}$$

where $h_{yt} = (h_{t-1}, x_t(h_{t-1}), \delta_t(h_{t-1}, x_t(h_{t-1})), s_t)$. Notice that $x_t(h_{t-1})$ shows up on both sides of equation (14), so we require that the optimal choice $x_t(h_{t-1})$ satisfy a *fixed point property*. To get some intuition for this property, suppose that each sticky-price producer believes that all other sticky-price producers will choose some value, say, $\hat{x}_t$. This choice, together with the central bank's strategy and the inflation and output rules, induces the outcomes $\pi_t(\hat{h}_{yt})$ and $y_t(\hat{h}_{yt})$, where $\hat{h}_{yt} = (h_{t-1}, \hat{x}_t, \delta_t(h_{t-1}, \hat{x}_t), s_t)$. The fixed point property requires that for $\hat{x}_t$ to be part of an equilibrium, each sticky-price producer's best response must coincide with $\hat{x}_t$.

The basic idea behind our sophisticated policy construction is that the central bank starts by picking any desired competitive equilibrium allocations and sets its policy on the equilibrium path consistent with them. The central bank then constructs its policy off the equilibrium path so that even if an individual agent believes that all other agents will deviate to some specific action, that individual agent finds it optimal to choose a different action. In this sense, the policies are specified so that the fixed point property is satisfied at only the desired allocations.

We now analyze several possible ways for a central bank to attempt the implementation of competitive equilibria in which it uses interest rates as its monetary policy instrument.

With reversion to a money regime. We show first that in the simple sticky-price model, any competitive equilibrium in which the central bank uses the interest rate as its instrument in all periods can be uniquely implemented with sophisticated policies that involve a *one-period reversion to money*. Under these policies, after a deviation, the central bank switches to a money regime for one period.

More precisely, fix a desired competitive equilibrium outcome path $(x_t^*(s^{t-1}),\ \pi_t^*(s^t),\ y_t^*(s^t))$ together with central bank policies $i_t^*(s^{t-1})$. Consider the following trigger-type policy: If sticky-price producers choose x_t in period t to coincide with the desired outcomes $x_t^*(s^{t-1})$, then let central bank policy in t be $i_t^*(s^{t-1})$. If not, and these producers deviate to some $\hat{x}_t \neq x_t^*(s^{t-1})$, then for that period t, let the central bank switch to a money regime with a suitably chosen level of money growth. This level of money growth makes it not optimal for any individual sticky–price setter to cooperate with the deviation. If such a level of money growth exists, we say that the best responses of the sticky–price setters are *controllable*. The following lemma shows that this property holds for our model.

LEMMA 2 (Controllability of Best Responses with One-Period Price-Setting). For any history $(h_{t-1}, \hat{x}_t)$, if the central bank chooses the money regime, then there exists a choice for money growth μ_t such that

$$\hat{x}_t \neq E[\pi_t(\hat{h}_{yt}) + \gamma y_t(\hat{h}_{yt})], \tag{15}$$

where $h_{yt} = (h_{t-1}, \hat{x}_t, M, \mu_t)$.

Proof. Substituting (2) into (10), we have a result showing that if the central bank chooses the money regime with money growth μ_t, then output y_t and inflation π_t are uniquely determined and given by

$$y_t = \frac{\mu_t + \nu_t + y_{t-1} - \hat{x}_t}{1+\kappa}, \tag{16}$$

$$\pi_t = \kappa y_t + \hat{x}_t. \tag{17}$$

Hence,

$$E[\pi_t(\hat{h}_{yt}) + \gamma y_t(\hat{h}_{yt})] = \frac{\kappa + \gamma}{1+\kappa}(\mu_t + y_{t-1} - \hat{x}_t) + \hat{x}_t.$$

Clearly, then, any choice of $\mu_t \neq \hat{x}_t - y_{t-1}$ will ensure that (15) holds. QED

We use this lemma to guide our choice of the suitable money growth rate after deviations. We choose this growth rate to generate the same expected inflation as in the original equilibrium. (Of course, we could have chosen many other values that also would discourage deviations, but we found this value to be the most intuitive.[4]) In particular, if the producers deviate to some $\hat{x}_t \neq x_t^*(s^{t-1})$, then for that period t, let the central bank switch to a money regime with money growth set so that

$$\mu_t = \hat{x}_t - y_{t-1} + \frac{1+\kappa}{\kappa}\left[x_t^*(s^{t-1}) - \hat{x}_t)\right]. \tag{18}$$

Note that $\mu_t \neq \hat{x}_t - y_{t-1}$. With such a money growth rate, expected inflation is the same in the reversion period as it would have been in the desired outcome. From Lemma 1, such a choice of $\hat{x}_t$ cannot be part of an equilibrium. It is also easy to see that if a deviation occurs in period t, the economy returns to the desired outcomes in period $t+1$. We have established the following proposition.

PROPOSITION 1 (Unique Implementation with Money Reversion). Any competitive equilibrium outcome in which the central bank uses interest rates as its instrument can be implemented as a unique equilibrium with sophisticated policies with one-period reversion to a money regime. Moreover, under this rule, after any deviation in period t, the equilibrium outcomes from period $t+1$ are the desired outcomes.

A simple way to describe our unique implementation result is that controllability of best responses under some regime guarantees unique implementation of any desired outcome. We obtain controllability by reversion to a money regime. Note that even though the money regime is not used on the equilibrium path, it is useful as an off-equilibrium commitment that helps support

4. We choose this part of the policy as a clear demonstration that after a deviation, the central bank is not doing anything exotic, such as producing a hyperinflation. Rather, in an intuitive sense, the central bank is simply getting the economy back on the track it had been on before the deviation threatened to shift it in another direction.

desired outcomes in which the central bank uses interest rates on the equilibrium path.

Notice also that the proposition implies that deviations lead to only very transitory departures from desired outcomes. In particular, we do not achieve implementation by threatening the economy with dire outcomes after deviations. (Note that the particular result, that the economy returns exactly to the desired outcomes in the period after the deviation, would not hold in a version of this model with state variables, such as capital.)

So far we have focused on uniquely implementing competitive outcomes when the central bank uses interest rates as its instrument. Equations (16) and (17) imply that the equilibrium outcome under a money regime is unique, so that implementing desired outcomes is trivial when the central bank uses money as its instrument. Clearly, we can use a simple generalization of Proposition 1 to uniquely implement a competitive equilibrium in which the central bank uses interest rates in some periods and money in others.

With pure interest-rate rules. Now, as a second possible way for a central bank to implement competitive equilibria, we analyze pure interest-rate rules. We find that this way cannot achieve unique implementation.

We begin with a pure interest-rate rule of the form

$$i_t(s^{t-1}) = i_t^*(s^{t-1}) + \phi(x_t(s^{t-1}) - x_t^*(s^{t-1})), \tag{19}$$

where $i_t^*(s^{t-1})$ and $x_t^*(s^{t-1})$ are the interest rates and the sticky-price producer choices associated with a competitive equilibrium that the central bank wants to implement uniquely, and the parameter ϕ represents how aggressively the central bank changes interest-rates when private agents deviate from the desired equilibrium. Notice that this rule (19) specifies policy both on and off the equilibrium path. On the equilibrium path, $x_t(s^{t-1}) = x_t^*(s^{t-1})$, and the rule yields $i_t(s^{t-1}) = i_t^*(s^{t-1})$. Off the equilibrium path, the rule specifies how $i_t(s^{t-1})$ should differ from $i_t^*(s^{t-1})$ when $x_t(s^{t-1})$ differs from $x_t^*(s^{t-1})$. Pure interest-rate rules of the form (19) have been discussed by King (2000) and Svensson and Woodford (2005). We follow Cochrane (2007) and call (19) the *King rule*.

Note from Lemma 1 that $x_t(s^{t-1}) = E[\pi_t(s^t) \mid s^{t-1}]$, so that the King rule can be thought of as targeting expected inflation, in the sense that (19) is equivalent to

$$i_t(s^{t-1}) = i_t^*(s^{t-1}) + \phi(E[\pi_t(s^t) \mid s^{t-1}] - E[\pi_t^*(s^t) \mid s^{t-1}]). \tag{20}$$

We now show that if the central bank follows the King rule (19), it cannot ensure unique implementation of the desired outcome. Indeed, under this rule, the economy has a continuum of equilibria. More formally:

PROPOSITION 2 (Indeterminacy of Equilibrium under the King Rule). Suppose the central bank sets interest rates i_t according to the simple economy's King rule (19). Then any of the continuum of sequences indexed by the initial condition x_0 and the parameter c that satisfies

$$x_{t+1} = i_t + c\eta_t, \quad \pi_t = x_t + \kappa(1 + \psi c)\eta_t, \quad \text{and} \quad y_t = (1 + \psi c)\eta_t \tag{21}$$

is a sophisticated outcome.

Proof. In order to verify that the multiple outcomes that satisfy (21) are part of a period-zero competitive equilibrium, we need to check that they satisfy (1), (9), and (10). That they satisfy (9) follows by taking expectations of the second and third equations in (21). Substituting for i_t from (19) and for x_{t+1} from (21) into (1), we obtain that $y_t = (1 + \psi c)\eta_t$, as required by (21). Inspecting the expressions for π_t and y_t in (21) shows that they satisfy (10). Clearly, any such period-zero competitive equilibrium can be supported by a government strategy, σ_g, of the King rule form and appropriately chosen σ_x, σ_y, and σ_π. QED

The intuitive idea behind the multiplicity of equilibria associated with the initial condition x_0 is that interest-rate rules, including the King rule, induce nominal indeterminacy and do not pin down the initial price level. The intuitive idea behind the multiplicity of stochastic equilibria associated with $c \neq 0$ is that interest rates pin down only expected inflation and not the state-by-state realizations indexed by the parameter c.

Note that Proposition 2 implies that even if the King rule parameter $\phi > 1$, the economy has a continuum of equilibria. In that case, all but one of the equilibria has exploding inflation, in the sense that inflation eventually becomes unbounded. In the literature, researchers often restrict attention to bounded equilibria. We argue that, in this model, equilibria with exploding inflation cannot be dismissed on logical grounds. Indeed, these equilibria are perfectly reasonable because the inflation explosion is associated with a money supply explosion.

To see this association, suppose that the economy has no stochastic shocks and the desired outcomes are $\pi_t = 0$ and $y_t = 0$ in all periods. Then, from the cash-in-advance constraint (2), we know that the growth of the money supply is given by

$$\mu_t = x_t = \phi^t x_0. \tag{22}$$

Thus, in these equilibria, inflation explodes because money growth explodes. Each equilibrium is indexed by a different initial value of the endogenous variable x_0. This endogenous variable depends solely on expectations of future policy and is not pinned down by any initial condition or transversality condition.

Such equilibria are reasonable because at the core of most monetary models is the idea that the central bank's printing of money at an ever-increasing rate leads to a hyperinflation. In these equilibria, inflation does not arise from the speculative reasons analyzed by Obstfeld and Rogoff (1983), but from the conventional money-printing reasons analyzed by Cagan (1956). In this sense, our model predicts, for perfectly standard and sensible reasons, that the economy can suffer from any one of a continuum of very undesirable paths for inflation. (Cochrane [2007] makes a similar point for a flexible-price model.)

The same proposition obviously applies to more general interest-rate rules that are restricted to be the same on and off the equilibrium path. For example, Proposition 2 applies to linear feedback rules of the form

$$i_t = \bar{\imath}_t + \sum_{s=0}^{\infty} \phi_{xs} x_{t-s} + \sum_{s=1}^{\infty} \phi_{ys} y_{t-s} + \sum_{s=1}^{\infty} \phi_{\pi s} \pi_{t-s}, \tag{23}$$

where the intercept term $\bar{\imath}_t$ can depend on the history of stochastic events.

With reversion to a hybrid rule. Analysis of a third possible way to implement competitive equilibria is a bit more complicated. In Proposition 1, we have shown how reversion to a money regime can achieve unique implementation. In Proposition 2 and the subsequent discussion, we have shown that pure interest-rate rules, such as the King rule, cannot. Notice that in our money reversion policies, even tiny deviations trigger a reversion to a money regime. A natural question arises: Can unique implementation be achieved using a combination of these two strategies, or a *hybrid rule,* specifying, for example, that the central bank continue to use interest rates unless the deviations are very large and then revert to a money regime? The answer is yes.

To see this, consider a particular hybrid rule that is intended to implement a bounded competitive equilibrium $\{x_t^*(s^{t-1}),\ \pi_t^*(s^t),\ y_t^*(s^t)\}$ with an associated interest rate $i_t^*(s^{t-1})$. Fix some $\bar{x}$ and $\underline{x}$ which satisfy $\bar{x} > \max_t x_t^*(s^{t-1})$ and $\underline{x} < \min_t x_t^*(s^{t-1})$. What we will call the *King–money hybrid rule* specifies that if $x_t(s^{t-1})$ is within the interest-rate interval $[\underline{x}, \bar{x}]$, then the central bank follows a King rule of the form (19); and if $x_t(s^{t-1})$ falls outside this interval, then the central bank reverts to a money regime and chooses the money growth rate that produces an expected inflation rate $\bar{\pi} \in [\underline{x}, \bar{x}]$. That the money growth rate can be so chosen follows from (16) and (17).

We show that an attractive feature of outcomes under this hybrid rule is that deviations from the desired path lead only to very transitory movements away from the desired path. More precisely, after any deviation in period t, even though inflation and output in period t may differ from the desired outcomes, those in subsequent periods coincide with the desired outcomes. More formally:

PROPOSITION 3 (Unique Implementation with a Hybrid Rule). In the simple economy, the King–money hybrid rule with $\phi > 1$ uniquely implements any bounded competitive equilibrium. Moreover, under this rule, after any deviation in period t, the equilibrium outcomes from period $t+1$ are the desired outcomes.

We prove this proposition in the Appendix. Here we simply sketch the argument for a deterministic version of the model. The key to the proof is a preliminary result that shows that no equilibrium outcome x_t can be outside the interval $[\underline{x}, \bar{x}]$. To see that this is true, suppose that in some period t, x_t is outside that interval. But when this is true, the hybrid rule specifies a money growth rate in that period that yields expected inflation inside the interval. Because x_t equals expected inflation, this gives a contradiction and proves the preliminary result.

To establish uniqueness, suppose that there is some sophisticated equilibrium with $\hat{x}_r \neq x_r^*$ for some r. From the preliminary result, $\hat{x}_r$ must be in the interval $[\underline{x}, \bar{x}]$ where the King rule is operative. From Lemma 1, we know that in any equilibrium, $i_t = x_{t+1}$, so that the King rule implies that

$$\hat{x}_{t+1} - x^*_{t+1} = \phi\left(\hat{x}_t - x^*_t\right) = \phi^{t-r}(\hat{x}_r - x^*_r).$$

Because $\phi > 1$ and x^*_t is bounded, eventually $\hat{x}_{t+1}$ must leave the interval $[\underline{x}, \bar{x}]$, which is a contradiction.

Extension to Interest-Elastic Money Demand. So far, to keep the exposition simple, we have assumed a cash-in-advance setup in which money demand is interest-inelastic. This feature of the model implies that if a money regime is adopted in some period t, then the equilibrium outcomes in that period are uniquely determined by the money growth rate in that period. This uniqueness under a money regime is what allows the central bank to switch to a one-period money regime in order to support any desired competitive equilibrium. Now we consider economies with interest-elastic money demand. We argue that under appropriate conditions, our unique implementation result extends to such economies.

When economies have interest-elastic money demand, sophisticated policies that specify reversion to money or to a hybrid rule can uniquely implement any desired outcome if best responses are *controllable*. A sufficient condition for such controllability is that competitive equilibria are unique under a suitably chosen money regime. Here, as with inelastic money demand, the uniqueness under a money regime is what enables unique implementation.

A sizable literature has analyzed the uniqueness of competitive equilibria under money growth policies with interest-elastic money demand. Obstfeld and Rogoff (1983) and Woodford (1994) provide sufficient conditions for this uniqueness. For example, Obstfeld and Rogoff (1983) consider a money-in-the-utility-function model with preferences of the form $u(c) + v(m)$, where c is consumption and m is real money balances, and show that a sufficient condition for uniqueness under a money regime is

$$\lim_{m \to 0} mv'(m) > 0.$$

Obstfeld and Rogoff (1983) focus attention on flexible-price models, but their results can be readily extended to our simple sticky-price model. Indeed, their sufficient conditions apply unchanged to a deterministic version of that model because our model without shocks is effectively identical to a flexible-price model. Hence, under appropriate sufficient conditions, our unique implementation result extends to environments with interest-elastic money demand.

More generally, for our hybrid rule to uniquely implement desired outcomes, we need a reversion policy that has a unique equilibrium. An alternative to a money regime is a commodity standard such as those in the work of Wallace (1981) and Obstfeld and Rogoff (1983). With this type of standard, the government promises to redeem money for goods for some arbitrarily low price and finances the supply of goods with taxation. An alternative to our hybrid rule with money reversion is, therefore, a hybrid rule with reversion to a commodity standard.

III. A Model with Staggered Price-Setting

We turn now to a version of our simple model with staggered price-setting, often referred to as the *New Keynesian model*. We show that, along the lines of the argument developed above, policies with infinite reversion to either a money regime or a hybrid rule can uniquely implement any desired outcome under an interest-rate regime. We also show that for a large class of economies, pure interest-rate rules of the King form still lead to indeterminacy. To make our points in the simplest way, we abstract from aggregate uncertainty.

III.A. Setup and Competitive Equilibrium

We begin by setting up the model with staggered price-setting. In the model, prices are set in a staggered fashion as in the work of Calvo (1983). At the beginning of each period, a fraction $1 - \alpha$ of producers are randomly chosen and allowed to reset their prices. After that, the central bank makes its decisions, and then, finally, consumers make theirs. This economy has no flexible-price producers.

The linearized equations in this model are similar to those in the simple model. The Euler equation (1) and the quantity equation (2) are unchanged, except that here they have no shocks. The price set by a producer permitted to reset its price is given by the analog of (4), which is

$$p_{st}(j) = (1 - \alpha\beta)\left[\sum_{r=0}^{\infty}(\alpha\beta)^{r-t}(\gamma y_r + p_r)\right], \tag{24}$$

where β is the discount factor. Here, again, Taylor's γ is the elasticity of the equilibrium real wage with respect to output. Letting p_{st} denote the average price set by producers permitted to reset their prices in period t, we can recursively rewrite this equation as

$$p_{st}(j) = (1-\alpha\beta)(\gamma y_t + p_t) + \alpha\beta p_{st+1}, \tag{25}$$

together with a type of transversality condition $\lim_{T\to\infty}(\alpha\beta)^T p_{sT}(j) = 0$. The aggregate price level can then be written as

$$p_t = \alpha p_{t-1} + (1-\alpha)p_{st}. \tag{26}$$

To make our analysis parallel to the literature, we again translate the decisions of the sticky-price producers from price levels to inflation rates. Letting $x_t(j) = p_{st}(j) - p_{t-1}$ and letting x_t denote the average of $x_t(j)$, with some manipulation we can rewrite (25) as

$$x_t = (1-\alpha\beta)\gamma y_t + \pi_t + \alpha\beta x_{t+1}. \tag{27}$$

We can also rewrite (26) as

$$\pi_t = (1-\alpha)x_t \tag{28}$$

and the transversality condition as $\lim_{T\to\infty}(\alpha\beta)^T x_t(j) = 0$. Using (28) and the fact that x_t is the average of $x_t(j)$ implies this condition is equivalent to

$$\lim_{t\to\infty}(\alpha\beta)^t \pi_t = 0. \tag{29}$$

In addition to these conditions, we now argue that in this staggered price-setting model, a competitive equilibrium must satisfy two boundedness conditions. In general, boundedness conditions are controversial in the literature. Standard analyses of New Keynesian models impose strict boundedness conditions: in any reasonable equilibrium, both output and inflation must be bounded both above and below. Cochrane (2007) has forcefully criticized this practice, arguing that any boundedness condition must have a solid economic rationale.

Here we provide rationales for two such conditions: output y_t must be bounded above, so that

$$y_t \le \bar{y} \quad \text{for some } \bar{y}, \tag{30}$$

and interest rates must be bounded below, so that

$$i_t \geq \underline{i} \quad \text{for some } \underline{i}. \tag{31}$$

The rationale for output being bounded above is that the economy has a finite amount of labor to produce the output. The rationale for requiring that interest rates be bounded below comes from the restriction that the nominal interest rate must be nonnegative.[5] These bounds allow outcomes in which (the log of) output, y_t, falls without bound (so that the level of output converges to zero). The bounds also allow for outcomes in which inflation rates explode upward without limit.

Here, then, a collection of allocations, prices, and policies $a_t = \{x_t, \delta_t, \pi_t, y_t\}$ is a *competitive equilibrium* if it satisfies (i) consumer optimality, namely, the deterministic versions of (1) and (2); (ii) sticky-price producer optimality, (27)–(29); and (iii) the boundedness conditions, (30) and (31).

Note that any allocations that satisfy (27)–(29) also satisfy the New Keynesian Phillips curve,

$$\pi_t = \kappa y_t + \beta \pi_{t+1}, \tag{32}$$

where now $\kappa = (1 - \alpha)(1 - \alpha\beta)\gamma/\alpha$. To see this result, use (28) to substitute for x_t and x_{t+1} in (27) and collect terms.

Here, as we did in the simple-sticky price model, we define *continuation competitive equilibria*. For example, consider the beginning of period t with a state variable y_{t-1}. A collection of allocations $a(y_{t-1}) = \{x_r(y_{t-1}), \delta_r(y_{t-1}), \pi_r(y_{t-1}), y_r(y_{t-1})\}_{r \geq t}$ is a continuation competitive equilibrium with y_{t-1} if it satisfies the three conditions of a competitive equilibrium above in all periods $r \geq t$. A continuation competitive equilibrium that starts at the end of period t given (y_{t-1}, x_t, δ_t) is defined similarly. This definition requires optimality by consumers from t onward and optimality by sticky-price producers from $t + 1$ onward.

III.B. Sophisticated Equilibrium

We turn now to sophisticated equilibrium in the staggered price-setting model, its definition and how it can be implemented.

5. Note that even though the real value of consumer holdings of bonds must satisfy a transversality condition, this condition does not impose any restrictions on the paths of y_t and π_t. The reason is that in our nonlinear model, the government has access to lump-sum taxes, so that government debt can be arbitrarily chosen to satisfy any transversality condition.

Definition. The definition of a *sophisticated equilibrium* in the staggered price-setting model parallels that in the simple sticky-price model. The elements needed for that definition are basically the same. The public events that occur in a period are, in chronological order, $q_t = (x_t; \delta_t; y_t, \pi_t)$. We let h_{t-1} denote the history of these events up until the beginning of period t. A strategy for the sticky-price producers is a sequence of rules $\sigma_x = \{x_t(h_{t-1})\}$. The public history faced by the central bank is $h_{gt} = (h_{t-1}, x_t)$ and its strategy, $\{\delta_t(h_{gt})\}$. The public history faced by consumers in period t is $h_{yt} = (h_{t-1}, x_t, \delta_t)$. We let $\sigma_y = \{y_t(h_{yt})\}$ and $\sigma_\pi = \{\pi_t(h_{yt})\}$ denote the sequences of output and inflation rules. Strategies and allocation rules induce continuation outcomes written as $\{a_r(h_{t-1}; \sigma)\}_{r \geq t}$ or $\{a(h_{yt}; \sigma)\}_{r \geq t}$ in the obvious recursive fashion.

Formally, then, a *sophisticated equilibrium* given the policies here is a collection of strategies (σ_x, σ_g) and allocation rules (σ_y, σ_π) such that (i) given any history h_{t-1}, the continuation outcomes $\{a_r(h_{t-1}; \sigma)\}_{r \geq t}$ induced by σ constitute a continuation competitive equilibrium and (ii) given any history h_{yt}, so do the continuation outcomes $\{a_r(h_{yt}; \sigma)\}_{r \geq t}$.

In this model, as in the simple sticky-price model, the choices of the sticky-price producers must satisfy a key fixed point property, that

$$x_t(h_{t-1}) = (1 - \alpha\beta)\gamma y_t(h_{yt}) + \pi_t(h_{yt}) + \alpha\beta x_{t+1}(h_t), \tag{33}$$

where $h_{yt} = (h_{t-1}, x_t(h_{t-1}), \delta_t(h_{t-1}, x_t(h_{t-1})))$ and $h_t = (h_{yt}, \pi_t(h_{yt}), y_t(h_{yt}))$. Here, as in the simple sticky-price model, $x_t(h_{t-1})$ shows up on both sides of the fixed point equation—on the right side, through its effect on the histories h_{yt} and h_t.

Implementation with Sophisticated Policies. We now show that in the staggered price-setting model, any competitive equilibrium can be uniquely implemented with sophisticated policies.

The basic idea behind our construction is, again, that the central bank starts by picking any competitive equilibrium allocations and sets its policy on the equilibrium path consistent with those allocations. The central bank then constructs its policy off the equilibrium path so that any deviations from these allocations would never be a best response for any individual price-setter. In so doing, the constructed sophisticated policies support the chosen allocations as the unique equilibrium allocations.

As we did with the simple model, here we show that, under sufficient conditions, policies that specify infinite reversion to a money regime can achieve unique implementation, a pure interest-rate rule of the King rule form cannot, and a King–money hybrid rule can.

With reversion to a money regime. We start with sophisticated policies that specify reversion to a money regime after deviations. In our construction of sophisticated policies, we assume that the best responses of sticky-price producers are *controllable* in that if they deviate by setting $\hat{x}_t \neq x_t^*$, then by infinitely reverting to the money regime, the central bank can set money growth rate policies so that the profit-maximizing value of $x_t(j)$ is such that $x_t(j) \neq \hat{x}_t$.

The sophisticated policy that supports a desired outcome is to follow the chosen monetary policy as long as private agents have not deviated from the desired outcome. If sticky-price producers ever deviate to some choice $\hat{x}_t$, the central bank switches to a money regime set such that $x_t(j) \neq \hat{x}_t$. The following proposition follows immediately:

PROPOSITION 4 (Unique Implementation with Money Reversion). If the best responses of the sticky-price producers are controllable, then any competitive equilibrium outcome in which the central bank uses interest rates as its instrument can be implemented as a unique equilibrium by sophisticated policies which specify reversion to a money regime.

A sufficient condition for best responses to be controllable is that in the nonlinear economy, preferences are given by $U(c,l) = \log c + b(1-l)$, where c is consumption and l is labor supply, so that in the linearized economy, Taylor's γ equals one. To demonstrate controllability, suppose that after a deviation, the central bank reverts to a constant money supply $m = \log M$. With a constant money supply, it is convenient to use the original formulation of the economy with price levels rather than inflation rates. With that translation, the cash-in-advance constraint implies that $y_r + p_r = m$ for all r, so that (24) implies that the producer's price is simply to set

$$p_{st}(j) = (1-\alpha\beta)\left[\sum_{r=0}^{\infty}(\alpha\beta)^{r-t}m\right] = m. \tag{34}$$

That is, if after a deviation the central bank chooses a constant level of the money supply m, then sticky-price producers optimally choose their prices to be m. Clearly, (34) implies that the best responses of these producers are controllable. For example, consider a history in which price-setters in period t deviate from p^*_{st} to $\hat{p}_{st}$. Obviously, the central bank can choose the level of the money supply so that the optimal choice for an individual price-setter becomes $p_{st}(j) \neq \hat{p}_{st}$, so that $x_t(j) = m - p_{t-1} \neq \hat{x}_t$.

With pure interest-rate rules. Now, as with the simple model, we turn to pure interest-rate rules such as the King rule. For the staggered price-setting model, we ask, can such rules uniquely implement bounded competitive equilibrium? We find that for a large class of parameter values, the answer is, again, no.

We arrive at this answer by first showing that under the King rule, the economy has a continuum of period-zero competitive equilibria. We then argue that associated with each competitive equilibrium is a sophisticated equilibrium.

Here, we write the King rule as

$$i_t = i^*_t + \phi(1-\alpha)(x_t - x^*_t), \tag{35}$$

where i^*_t and π^*_t are the interest rates and the inflation rates associated with the desired (bounded) competitive equilibrium. From (28), it follows that in all periods, inflation and the aggregate price-setting choice are mechanically linked by $\pi_t = (1-\alpha)x_t$. This mechanical link means that we can equally well think of policy as feeding back on either inflation or the price-setting choice, so that (35) is equivalent to

$$i_t = i^*_t + \phi(\pi_t - \pi^*_t). \tag{36}$$

Now we show that the economy has a continuum of competitive equilibria by showing that there is a continuum of solutions to (1), (32), and (36) and that these solutions do not violate the transversality and boundedness conditions (29), (30), and (31).

Expressing the variables as deviations from the desired equilibrium is convenient. To that end, let $\tilde{\pi}_t = \pi_t - \pi^*_t$ and $\tilde{y}_t = y_t - y^*_t$. Subtracting the equations governing $\{\pi^*_t, y^*_t\}$ from those governing $\{\pi_t, y_t\}$ gives a system governing $\{\tilde{\pi}_t, \tilde{y}_t\}$ that satisfies (1), (32), and (36). Substituting for $\tilde{\imath}_t$ in (1), using (36), we get that

$$\tilde{y}_{t+1} + \psi\tilde{\pi}_{t+1} = \tilde{y}_t + \psi\phi\tilde{\pi}_t, \tag{37}$$

and from (32) we have that

$$\tilde{\pi}_t = \kappa \tilde{y}_t + \beta \tilde{\pi}_{t+1}. \tag{38}$$

Equations (37) and (38) define a dynamical system. Letting $z_t = (\tilde{y}_t, \tilde{\pi}_t)'$, with some manipulation we can stack these equations to give $z_{t+1} = Az_t$, where

$$A = \begin{bmatrix} a & b \\ \frac{-\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

and where $a = 1 + \kappa\psi/\beta$ and $b = \psi(\phi - 1/\beta)$. This system has a continuum of solutions of the form

$$\begin{aligned} \tilde{y}_t &= \lambda_1^t \omega_1 + \lambda_2^t \omega_2 \quad \text{and} \\ \tilde{\pi}_t &= \lambda_1^t \left(\frac{\lambda_1 - a}{b}\right) \omega_1 + \lambda_2^t \left(\frac{\lambda_2 - a}{b}\right) \omega_2, \end{aligned} \tag{39}$$

where $\lambda_1 < \lambda_2$, the eigenvalues of A, are given by

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(\frac{1+\kappa\psi}{\beta} + 1\right) \pm \frac{1}{2}\sqrt{\left(\frac{1+\kappa\psi}{\beta} - 1\right)^2 - 4(\phi - 1)\frac{\kappa\psi}{\beta}}, \tag{40}$$

and $\omega_1 = [(\frac{\lambda_2 - a}{b})\tilde{y}_0 - \tilde{\pi}_0]/\Delta$ and $\omega_2 = [(\frac{a-\lambda_1}{b})\tilde{y}_0 + \tilde{\pi}_0]/\Delta$, where Δ is the determinant of A.[6] This continuum of solutions is indexed by $\tilde{y}_0$ and $\tilde{\pi}_0$.

In the Appendix, we show that for a class of economies that satisfy the restriction

$$1 - \kappa\psi < \beta \quad \text{and} \quad \alpha(1 + \kappa\psi) < 1, \tag{41}$$

equilibrium is indeterminate under the King rule. We can think of (41) as requiring that the period length is sufficiently short, in the sense that β is close enough to 1, and that the price stickiness is not too large, in the sense that α is sufficiently small. Formally, in the Appendix, we prove the following proposition:

PROPOSITION 5 (Indeterminacy of Equilibrium under the King Rule). Suppose that the central bank sets interest rates i_t according to the King rule (35) with $\phi > 1$ and that (41) is

6. Here and throughout, we restrict attention to values of $\phi \in [0, \phi_{\max}]$, where $\phi_{\max}$ is the largest value of ϕ that yields real eigenvalues. That is, at $\phi_{\max}$, the discriminant in (40) is zero.

satisfied. Then the economy has a continuum of competitive equilibria indexed by $y_0 \leq y_0^*$,

$$y_t = y_t^* + \lambda_2^t (y_0 - y_0^*) \quad \text{and} \quad \pi_t = \pi_t^* + \lambda_2^t c(y_0 - y_0^*), \tag{42}$$

where $\lambda_2 > 1$ and $c = (\lambda_2 - a)/b < 0$ are constants.

It is immediate to construct a sophisticated equilibrium for each of the continuum of competitive equilibria in (42).

Notice that under the King rule, there is one equilibrium with $y_t = y_t^*$ and $\pi_t = \pi_t^*$ for all t, and in the rest, y_t goes to minus infinity and π_t to plus infinity. All of these equilibria satisfy the boundedness conditions (30) and (31) and, under (41), the transversality condition (29).

It turns out that if the inequality in the second part of (41) is reversed, then the set of solutions to the New Keynesian dynamical system, (1), (28), (32), and (35), has the form (42), but the transversality condition rules out all solutions except the one with $y_t = y_t^*$ and $\pi_t = \pi_t^*$ for all t. We find this way of ruling out solutions unappealing because it hinges critically on the idea that sticky-price producers may be unable to change their prices for extremely long periods, even in the face of exploding inflation.

With reversion to a hybrid rule. We now show that in the staggered price-setting model, as in the simple model, a King–money hybrid rule can uniquely implement any bounded competitive equilibrium.

To do so in this model, we will assume *boundedness under money*, namely, that for any state variable y_{t-1} there exists a money regime from period t onward such that a continuation competitive equilibrium exists, and for all such equilibria, inflation in period t, π_t, is uniformly bounded. Here *uniformly bounded* means that there exist constants $\underline{\pi}$ and $\bar{\pi}$ such that for all y_{t-1}, $\pi_t \in [\underline{\pi}, \bar{\pi}]$. It is immediate that a sufficient condition for boundedness under money is that preferences in the nonlinear economy are given by $U(c, l) = \log c + b(1 - l)$.

In an economy that satisfies boundedness under money, the King–money hybrid rule that implements a competitive equilibrium $\{x_t^*, \pi_t^*, y_t^*\}$ with an associated interest rate i_t^* is defined as follows. Set $\bar{x}$ to be greater than both $\max_t x_t^*$ and $\bar{\pi}$, and set $\underline{x}$ to be lower than both $\min_t x_t^*$ and $\underline{\pi}$. This rule specifies that if $x_t \in [\underline{x}, \bar{x}]$, then the central bank follows a King rule of the form

(35) with $\phi > 1$. If x_t falls outside the interval $[\underline{x}, \bar{x}]$, then the central bank reverts to a money regime forever.

PROPOSITION 6 (Unique Implementation with a Hybrid Rule). *Suppose the staggered price-setting economy satisfies boundedness under money. Then the King–money hybrid rule implements any desired bounded competitive equilibrium. Moreover, under this rule, after any deviation in period t, the equilibrium outcomes from period $t+1$ are the desired outcomes.*

The formal proof of this proposition is in the Appendix. The key idea of this proof is the same as that for this proof of Proposition 3. The idea is that under the King rule, any $\hat{x}_t$ that does not equal x_t^* leads subsequent price-setting choices to eventually leave the interval $[\underline{x}, \bar{x}]$. But given boundedness under money, price-setting choices outside of the interval $[\underline{x}, \bar{x}]$ cannot be part of an equilibrium.

Note that with the staggered price-setting model, as with the simple model, under a hybrid rule, deviations lead to only very transitory departures from desired outcomes.

IV. Trembles and Imperfect Information

We have shown that in both of the models we have analyzed—a simple one-period price-setting model and a staggered price-setting model—any equilibrium outcome can be implemented as a unique equilibrium with sophisticated policies. In our equilibria, deviations in private actions lead to changes in the regime. This observation leads to the question of how to construct sophisticated policies if trembles in private actions occur or if deviations in private actions can be detected only imperfectly, say, with measurement error. We show that we can achieve unique implementation with trembles. We show that, with imperfect detection, the King–money hybrid rule leads to a unique equilibrium. This equilibrium is arbitrarily close to the desired equilibrium when the detection error is small. In this sense, our results are robust to trembles and imperfect information.

IV.A. Trembles

Unique implementation is not a problem if trembles in private actions occur.

To see that, consider allowing for trembles in private decisions by supposing that the actual price chosen by a price-setter, $x_t(j)$, differs from the intended price, $\tilde{x}_t(j)$, by an additive error $\varepsilon_t(j)$, so that $x_t(j) = \tilde{x}_t(j) + \varepsilon_t(j)$.

Trembles are clearly a trivial consideration. If $\varepsilon_t(j)$ is independently distributed across agents, then it simply washes out in the aggregate; it is irrelevant. Even if $\varepsilon_t(j)$ is correlated across agents, say, because it has both aggregate and idiosyncratic components, our argument goes through unchanged if the central bank can observe the aggregate component, for example, with a random sample of prices.

IV.B. Imperfect Information

Not as trivial is a situation in which the central bank has imperfect information about prices. But even in that situation, the King–money hybrid rule leads to a unique equilibrium; and when the detection error is small, this equilibrium is arbitrarily close to the desired equilibrium.

To see that, consider a formulation in which the central bank observes the actions of price-setters with measurement error. Of course, if the central bank could see some other variable perfectly, such as output or interest rates on private debt, then it could infer what the private agents did. We think of this formulation as giving the central bank minimal amounts of information relative to what actual central banks have. We show here that with this sort of imperfect information, we can implement outcomes that are close to the desired outcomes when the measurement error is small.

Here the central bank observes the price-setters' choices with error, so that

$$\hat{x}_t = x_t + \varepsilon_t, \tag{43}$$

where the error ε_t is i.i.d. over time with mean zero and bounded support $[\underline{\varepsilon}, \bar{\varepsilon}]$. Consider using the King–money hybrid rule to support some desired competitive equilibrium. Choose the interest-rate interval $[\underline{x}, \bar{x}]$ such that $x_t^* + \varepsilon_t$ is contained in this interval for all t. Here, the King rule is of the form

$$i_t(h_{gt}) = i_t^* + \phi(1-\alpha)(\hat{x}_t - x_t^*) \tag{44}$$

with $\phi > 1$.

In this economy with measurement error, the best response of any individual price-setter is identical to that in the economy without measurement error. This result follows because the best response depends on only the expected values of future variables. Because the measurement error ε_t has mean zero, these expected values are unchanged. Therefore, the unique equilibrium in this economy with measurement error has $x_t = x_t^*$; thus, $\pi_t = \pi_t^*$. The realized values of the interest rate i_t and output y_t, however, fluctuate around their desired values i_t^* and y_t^*. Using (43) and (44), we know that the realized value of the interest rate is given by

$$i_t = i_t^* + \phi(1-\alpha)\varepsilon_t, \tag{45}$$

whereas using the Euler equation, we know that the realized value of output is given by

$$y_t = y_t^* - \psi\phi(1-\alpha)\varepsilon_t. \tag{46}$$

Notice that when the central bank observes private actions imperfectly, the King–money hybrid rule does not exactly implement any desired competitive equilibrium. Rather, this rule implements an equilibrium in which output fluctuates around its desired level. These fluctuations are proportional to the size of the measurement error. Clearly, as the size of the measurement error ε_t goes to zero, the outcomes converge to the desired outcomes. We have thus established a proposition:

PROPOSITION 7 (Approximate Implementation with Measurement Error). Suppose the sophisticated policy is described by the King–money hybrid rule described above. Then the economy has a unique equilibrium with $x_t = x_t^*$ and y_t given by (46). As the variance of the measurement error approaches zero, the economy's outcomes converge to the desired outcomes.

Note that although the central bank never reverts to a money regime when it is on the equilibrium path, the possibility that it will do so off the equilibrium path plays a critical role in this implementation.

V. IMPLICATIONS FOR THE TAYLOR PRINCIPLE

The sophisticated policy approach we have just described has implications for the use of the Taylor principle as a device to ensure determinacy and to guide inferences from empirical investigations about whether central bank policy has led the economy into a determinate or indeterminate region. (Recall that the *Taylor principle* is the notion that interest rates should rise more than one for one with inflation rates, both compared to some exogenous, possibly stochastic, levels.)

V.A. Setup

In order to show what the sophisticated policy approach implies for our discussion of the Taylor principle, we consider a popular specification of the Taylor rule of the form

$$i_t = \bar{\imath}_t + \phi E_{t-1}\pi_t + bE_{t-1}y_t, \tag{47}$$

where $\bar{\imath}_t$ is an exogenously given, possibly stochastic, sequence. (See Taylor [1993] for a similar specification.) In our simple model, from (12), policies of the Taylor rule form (47) can be written as

$$i_t = \bar{\imath}_t + \phi(x_t - \bar{x}_t). \tag{48}$$

When the parameter $\phi > 1$, such policies are said to satisfy the Taylor principle: The central bank should raise its interest rate more than one for one with increases in inflation. When $\phi < 1$, such policies are said to violate that principle. Notice that when $\bar{\imath}_t$ and $\bar{x}_t$ coincide with the desired competitive equilibrium outcomes i_t^* and x_t^* for all periods, the Taylor rule (48) reduces to the simple model's King rule (19).

V.B. Implications for Determinacy

Many economists have argued that central banks must adhere to the Taylor principle in order to ensure unique implementation. Our results clearly imply that if the central bank is following a pure interest-rate rule, then adherence to the Taylor principle is neither necessary nor sufficient for unique implementation. If, however, the central bank is following a King–money hybrid rule, then adherence to this principle after deviations between observed outcomes and desired outcomes can help ensure unique implementation.

Note that policies of the Taylor rule form (48) are linear feedback rules of the form (23) and lead to indeterminacy, regardless of the value of ϕ. In this sense, if the central bank is following a pure interest-rate rule, then adherence to the Taylor principle is not sufficient for unique implementation. A similar argument implies that, under (41), it is not sufficient in the staggered price-setting model either.

Clearly, under pure interest-rate rules, adherence to the Taylor principle is also not necessary for unique implementation. Propositions 1 and 4 imply that, in both models, the central bank can uniquely implement any competitive equilibrium, including those that violate the Taylor principle along the equilibrium path.

V.C. Implications for Estimation

Many economists have estimated monetary policy rules and then inferred that these rules have led the economy to be in the determinate region if and only if they satisfy the Taylor principle. Indeed, one branch of this literature argues that the undesirable inflation experiences of the 1970s in the United States occurred in part because monetary policy led the economy to be in the indeterminate region. See, for example, the work of Clarida, Galí, and Gertler (2000).

We provide a set of stark assumptions under which such inferences can be made more confidently. Nonetheless, finding appropriate assumptions in more interesting applied examples remains a challenge.

Perfect Information. In economies in which the central bank and private agents have the same information, observations of variables along the equilibrium path shed no light on the properties of policies off that path, and it is these properties that govern the determinacy of equilibrium. Of course, any estimation procedure can rely only on data along the equilibrium path; it cannot uncover the properties of policies off that path. In this sense, estimation procedures in economies with perfect information cannot determine whether monetary policy is leading the economy to be in the determinate or the indeterminate region. (See Cochrane [2007] for a related point.)

To see this general point in the context of our models, note that any estimation procedure can only uncover relationships between the equilibrium interest rate i_t^* and the equilibrium inflation rate π_t^*. These relationships have nothing whatsoever to do with the off-equilibrium path policies that govern determinacy. For example, in the context of the King–money hybrid rule with the King rule of the form (35), neither i_t^* nor π_t^* depend on the parameter ϕ, but the size of this parameter plays a key role in ensuring determinacy. In this sense, without trivial identifying assumptions, no estimation procedure can uncover the key parameter for determinacy.

For example, suppose that along the equilibrium path, interest rates satisfy

$$i_t^* = \bar{\imath} + \phi^*(x_t^* - \bar{x}), \tag{49}$$

where i_t^* and x_t^* are the desired equilibrium outcomes and $\bar{\imath}$ and $\bar{x}$ are some constants that differ from those desired outcomes. This equilibrium can be supported in many ways, including reversion after deviations to a money regime or some sort of hybrid rule. Notice that in (49) the parameter ϕ^* simply describes the relation between the equilibrium outcomes i_t^* and x_t^* and has no connection to the behavior of policy after deviations.

Obviously, with a policy that specifies reversion to a money regime, the size of ϕ^* (whether it is smaller or larger than one) has no bearing on the determinacy of equilibrium.

That is also true with a policy that reverts to a hybrid rule after deviations, though perhaps not as obviously. Suppose that for small deviations, the hybrid rule specifies the King rule (20) with $\phi > 1$. The parameter ϕ of this King rule has no connection to the parameter ϕ^* in (49). The former governs the behavior of policies after deviations, whereas the latter simply describes a relationship that holds along the equilibrium path. Furthermore, although $\phi > 1$ ensures determinacy, the size of ϕ^*—whether it is smaller or larger than 1—has no bearing on determinacy.

These arguments clearly generalize to situations in which the constants $\bar{\imath}$ and $\bar{x}$ are replaced by exogenous, possibly stochastic, sequences $\bar{\imath}_t$ and $\bar{x}_t$ that differ from the desired outcomes, so that along the equilibrium path, interest rates satisfy

$$i_t^* = \bar{\imath}_t + \phi^*(x_t^* - \bar{x}_t). \tag{50}$$

We interpret most of the current estimation procedures of the Taylor rule variety as estimating ϕ^*, the parameter governing desired outcomes in (50) or its analog in more general setups. To use these estimates to draw inferences about determinacy, researchers implicitly assume that the parameter ϕ (the parameter describing off-equilibrium path behavior) is the same as ϕ^* (the parameter describing on-equilibrium path behavior). Researchers also restrict attention to bounded solutions. As we have discussed, with perfect information, theory imposes no connection between ϕ and ϕ^*, so the assumption that $\phi = \phi^*$ is not grounded in theory.

Also, the rationale for restricting attention to bounded solutions is not clear. With perfect information, then, current estimation procedures simply cannot uncover whether the economy is in the determinate or the indeterminate region.

Imperfect Information. With imperfect information, however, there is some hope that variants of current procedures may be able to uncover some of the key parameters for determinacy, provided researchers are willing to make some quite strong assumptions.

Here we provide a stark example in which a variant of current procedures can uncover one of the key parameters governing determinacy. Consider our staggered price-setting economy, in which the central bank observes the price-setters' choices with error. Recall that in this economy, the equilibrium outcomes for interest rates and output, (45) and (46), depend on the parameter ϕ in the King–money hybrid rule and that this parameter plays a key role in ensuring determinacy. Note the contrast with the perfect information economy, in which the equilibrium outcomes do not depend on the parameter ϕ. The fact that equilibrium outcomes depend on the key determinacy parameter here offers some hope that researchers will be able to estimate it.

For our stark example, we assume that researchers observe the same data as the central bank and that along the equilibrium path, the central bank follows a King rule of the form

$$i_t = i_t^* + \phi(1-\alpha)(\hat{x}_t - x_t^*). \tag{51}$$

If researchers know the desired outcomes x_t^* and i_t^*, as well as the parameter α, then they can simply solve (51) for ϕ as long as $\hat{x}_t$ does not identically equal x_t^*.

To go from this solution for ϕ to an inference about determinacy requires more assumptions. One set of assumptions is that the data are generated by our staggered price-setting model, in which the central bank observes $\hat{x}_t = x_t + \varepsilon_t$, where ε_t is i.i.d. over time and has mean zero and bounded support $[\underline{\varepsilon}, \bar{\varepsilon}]$, and the central bank follows the King–money hybrid rule, with the King rule given by (51). The key feature of the formulation that allows this inference is that $\hat{x}_t$ does not identically equal x_t^* as it does in the economies with perfect information.

Note that in our stark example, this procedure can uncover the King rule parameter ϕ, but not the hybrid rule parameters $\underline{\pi}$ and $\bar{\pi}$. More generally, no procedure can uncover what behavior would be in situations that are never reached in equilibrium, even if the specification of such behavior plays a critical role in unique implementation. This observation implies that even in our stark example, we cannot distinguish between a pure interest-rate rule and the King–money hybrid rule.

Although we have offered some hope for uncovering some of the key parameters for determinacy, applying our insight to a broader class of environments is apt to be hard. In practice, after all, the desired outcomes are not known, the other parameters of the economy are not known, the measurement error is likely to be serially correlated, and the interest-rate rule is subject to stochastic shocks.

Quite beyond these practical issues is a theoretical one: drawing inferences about determinacy requires confronting a subtle identification issue. This issue stems from the fact that characterizing the equilibrium is relatively easy if the economy is in the determinate region, but extremely hard if it is not. Specifically, if the economy is in the determinate region, then the probability distribution over observed variables is a relatively straightforward function of the primitive parameters. If the economy is in the indeterminate region, however, then this probability distribution (which must take account of the possibility of sunspots) is more complicated.

One way to proceed is to tentatively assume that the economy is in the determinate region and estimate the key parameters governing determinacy. Suppose that under this tentative assumption, we find that the parameters fall in the determinate region. Can we then conclude that the economy is in the determinate region? Not yet. We must still show that the data could not have been generated by one of the indeterminate equilibria—not an easy task.

VI. Conclusions

We have here described our sophisticated policy approach and illustrated its use as an operational guide to policy that achieves unique implementation of any competitive equilibrium outcome. We have demonstrated that using a pure interest-rate rule leads to indeterminacy. We have also constructed policies that avoid this by switching regimes: they use interest rates until private agents deviate and then revert to a money regime or a hybrid rule.

Our work has strong implications for the use of the Taylor principle as a guide to policy. We have shown that if a central bank follows a pure interest-rate rule, then adherence to the Taylor principle is neither necessary nor sufficient for unique implementation. Adherence to that principle may ensure determinacy, however, if monetary policy includes a reversion to the King–money hybrid rule after deviations.

We have also argued that existing empirical procedures used to draw inferences about the relationship between adherence to the Taylor principle and determinacy should be treated with caution. We have provided a set of stark assumptions that can be more confidently used in applied work to draw inferences regarding the relationship between central bank policy and determinacy. Using this method, however, requires solving multiple difficult identification problems.

Finally, although we have here focused exclusively on monetary policy, the use of our operational guide is not necessarily limited to that application. The logic behind the construction of the guide should be applicable as well to other governmental policies—for example, to fiscal policy and to policy responses to financial crises—or to any application that aims to uniquely implement a desired outcome.

Appendix: The Proofs of Propositions 3, 5, and 6

A. *Proof of Proposition 3: A Unique Implementation with a Hybrid Rule in the Simple Model*

Given that the central bank follows the King–money hybrid rule, say, σ_g^*, we will show here that there are unique strategies σ_x, σ_y, and σ_π for private agents that, together with σ_g^*, constitute a sophisticated equilibrium. We then show that this sophisticated equilibrium implements the desired outcomes.

The strategies σ_x, σ_y, and σ_π are as follows. The strategy σ_x specifies that $x_t(h_{t-1}) = x_t^*(s^{t-1})$ for all histories. The strategies σ_y and σ_π specify $y_t(h_{yt})$ and $\pi_t(h_{yt})$ as the unique solutions to conditions defining consumer optimality; (1) and (2), which define flexible price–producer optimality, (10); and the King–money hybrid rule with $y_{t+1}(s^{t+1}) = y_{t+1}^*(s^{t+1})$ and $x_{t+1}(s^{t+1}) = x_{t+1}^*(s^{t+1})$. Note that the value of x_t in the history $h_{yt} = (h_{t-1}, x_t, \delta_t, s_t)$ determines the regime in the current period and, hence, determines whether the Euler equation (1) or the cash-in-advance constraint (2) is used to solve for $y_t(h_{yt})$ and $\pi_t(h_{yt})$.

We now show that $(\sigma_g^*, \sigma_x, \sigma_y, \sigma_\pi)$ is a sophisticated equilibrium. Given that $\{x_t^*(s^{t-1}),\ \pi_t^*(s^t),\ y_t^*(s^t)\}$ is a period-zero competitive equilibrium and that $x_t^*(s^{t-1}) \in [\underline{x}, \bar{x}]$, so that the central bank is following an interest-rate regime, we know that any tail of these outcomes $\{x_t^*(s^{t-1}),\ \pi_t^*(s^t),\ y_t^*(s^t)\}_{t\geq r}$ is a continuation competitive equilibrium starting in period r regardless of the history h_{r-1}. On the equilibrium path, this claim follows immediately because the continuation of any competitive equilibrium is also a competitive equilibrium. Off the equilibrium path, for histories h_{t-1}, the tail is a period-zero competitive equilibrium (with periods suitably relabeled) and is, therefore, a continuation competitive equilibrium. A similar argument shows that the tail of the outcomes starting from the end of period r, namely, $\pi_r(h_{yr})$ and $y_r(h_{yr})$, together with the outcomes $\{x_t^*(s^{t-1}), \pi_t^*(s^t),\ y_t^*(s^t)\}_{t\geq r+1}$, constitutes a continuation competitive equilibrium.

Note that our construction implies that after any deviation in period t, the equilibrium outcomes from period $t+1$ are the desired outcomes.

We now establish uniqueness of the sophisticated equilibrium of the form $(\sigma_g^*, \sigma_x, \sigma_y, \sigma_\pi)$. We begin with a preliminary result that shows that for any s^{t-1} in any equilibrium, $x_t(s^{t-1}) \in [\underline{x}, \bar{x}]$. This argument is by contradiction. Suppose that at s^{t-1}, $x_t(s^{t-1}) \notin [\underline{x}, \bar{x}]$. Under the hybrid rule, the central bank reverts to a money regime with expected inflation equal to $\bar{\pi} \in [\underline{x}, \bar{x}]$. From Lemma 1, $x_t(s^{t-1}) = \bar{\pi} \in [\underline{x}, \bar{x}]$, which contradicts $x_t(s^{t-1}) \notin [\underline{x}, \bar{x}]$. This result implies that along the equilibrium path, the central bank never reverts to money, so that interest rates are given by the King rule (19).

With this preliminary result, we establish uniqueness by another contradiction argument. Suppose that the economy has a sophisticated equilibrium in which in some history h_{r-1}, $x_r(h_{r-1}) = \hat{x}_r$, which differs from $x_r^*(s^{r-1})$. Without loss of generality, suppose that $\hat{x}_r - x_r^*(s^{r-1}) = \varepsilon > 0$. Let $\{\hat{x}_t(s^{t-1}),\ \hat{\pi}_t(s^t),\ \hat{y}_t(s^t)\}_{t\geq r}$ denote the associated continuation competitive equilibrium outcomes. Our preliminary result implies that the central bank follows the King rule in all periods. Let $\{\hat{\imath}_t(s^{t-1})\}_{t\geq r}$ denote the associated interest rates. From (13), using the law of iterated expectations, we have that

$$E[i_t^*(s^{t-1}) \mid s^{r-1}] = E[x_{t+1}^*(s^t) \mid s^{r-1}] \quad \text{and}$$
$$E[\hat{\imath}_t(s^{t-1}) \mid s^{r-1}] = E[\hat{x}_{t+1}(s^t) \mid s^{r-1}]. \tag{52}$$

Substituting (52) into the King rule (19) gives that

$$E\big[\hat{x}_{t+1}(s^t) - x^*_{t+1}(s^t) \mid s^{r-1}\big] = \phi^{t-r}\varepsilon.$$

Because $\phi > 1$ and $x^*_{t+1}(s^t)$ is bounded, for every ε there exists some T such that

$$E\big[\hat{x}_{T+1}(s^T) \mid s^{T-1}\big] > \bar{x}.$$

But this contradicts our preliminary result that $x_t(s^{t-1}) \leq \bar{x}$ for all t and s^{t-1}. QED

B. Proof of Proposition 5: Indeterminacy of Equilibrium under the King Rule in the Staggered Price-Setting Model

It is straightforward to verify that output and inflation satisfying (42) satisfy all equilibrium conditions except the model's transversality condition (29) and its two boundedness conditions (30) and (31). Here we verify these conditions.

Consider first the transversality condition. Under (40) it follows that the larger eigenvalue $\lambda_2(\phi)$ is a decreasing function of ϕ and that $\lambda_2(1) = (1 + \kappa\psi)/\beta$. From (41) it then follows that $\beta\alpha\lambda_2(\phi) < 1$ for all $\phi \geq 1$. Hence, $\lim_{t\to\infty}(\alpha\beta)^t\tilde{\pi}_t = 0$. Because π^*_t is bounded, it follows that π_t satisfies the transversality condition (29).

Consider next the output and interest-rate boundedness conditions. We first show that $[\lambda_2(\phi) - a]/b < 0$ for all $\phi \geq 1$. To do so, we show that $\lambda_2(\phi) - a$ is positive for $\phi \in [1, 1/\beta)$, zero at $\phi = 1/\beta$, and negative for $\phi \in (1/\beta, \phi_{\max}]$. From (40) we know that

$$(53) \quad \lambda_2\left(\frac{1}{\beta}\right) = \frac{1}{2}\left(\frac{1+\kappa\psi}{\beta} + 1\right) + \frac{1}{2}\sqrt{\left[\left(\frac{1}{\beta} - 1\right) + \frac{\kappa\psi}{\beta}\right]^2 - 4\left(\frac{1}{\beta} - 1\right)\frac{\kappa\psi}{\beta}}.$$

Note that the term in the radical is a perfect square. Then using that and the first part of (41) turns (53) into

$$\lambda_2\left(\frac{1}{\beta}\right) = 1 + \frac{\kappa\psi}{\beta} = a.$$

Because $\lambda_2(\phi)$ is decreasing, it follows that $\lambda_2(\phi) - a$ has the desired sign pattern. Because $b = \psi(\phi - 1/\beta)$, the numerator and the denominator of $[\lambda_2(\phi) - a]/b$ have opposite signs for all $\phi \geq 1$, so that $[\lambda_2(\phi) - a]/b$ is negative. Thus, the boundedness conditions are satisfied for all $\omega_2 \leq 0$. In the resulting equilibria, inflation goes to plus infinity and output goes to minus infinity (so that the level of output goes to zero). QED

C. Proof of Proposition 6: Unique Implementation with a Hybrid Rule in the Staggered Price-Setting Model

Let $\{x_t^*,\ \pi_t^*,\ y_t^*\}$ be the desired bounded competitive equilibrium. The strategies that implement this competitive equilibrium are as follows. The strategy σ_g^* is the King–money hybrid rule. The strategy σ_x specifies that $x_t(h_{t-1}) = x_t^*$ for all histories. The strategies σ_y and σ_π specify $y_t(h_{yt})$ and $\pi_t(h_{yt})$ that are the unique solutions to the deterministic versions of the conditions defining consumer optimality, (1), (2), (28), (32), and the King–money hybrid rule with $y_{t+1} = y_{t+1}^*$ and $x_{t+1} = x_{t+1}^*$.

The proof that $(\sigma_g^*, \sigma_x, \sigma_y, \sigma_\pi)$ is a sophisticated equilibrium closely parallels that of Proposition 3.

We now establish uniqueness of the sophisticated equilibrium of the form $(\sigma_g^*, \sigma_x, \sigma_y, \sigma_\pi)$. We begin by showing that given σ_g^*, $x_t(h_{t-1}) = x_t^*$ for all histories. (Clearly, given σ_g^* and σ_x, σ_y and σ_π are unique.) For reasons similar to those underlying the preliminary result in Proposition 3, for any history h_{t-1}, $x_t(h_{t-1})$ must be in the interval $[\underline{x}, \bar{x}]$, so that for any history, interest rates are given by the King rule (35). Under an interest-rate rule, the state y_{t-1} is irrelevant; therefore, a continuation competitive equilibrium starting at the beginning of any period t solves the same equations as a competitive equilibrium (starting from period 0). For notational simplicity, we focus on a competitive equilibrium starting from period 0.

Suppose by way of contradiction that $\{\hat{x}_t,\ \hat{\pi}_t,\ \hat{y}_t\}$ is an equilibrium that does not coincide with $\{x_t^*,\ \pi_t^*,\ y_t^*\}$. Let $\tilde{x}_t = \hat{x}_t - x_t^*$, and use similar notation for $\tilde{\pi}_t$ and $\tilde{y}_t$. Then, subtracting the equations governing the systems denoted with an asterisk from those denoted with a caret, we have a system governing $\{\tilde{x}_t, \tilde{\pi}_t, \tilde{y}_t\}$ that satisfies (the analogs of) (1), (32), and (35). The resulting system, given by (37) and (38), coincides with that in the proof of Proposition 5. Hence, the solution is given by (39) with eigenvalues given by (40).

It is easy to check that $\phi > 1$ implies that both eigenvalues λ_1 and λ_2 are greater than one. Furthermore, at least one of $(\lambda_1 - a)/b$ and $(\lambda_2 - a)/b$ is nonzero. Because both of the eigenvalues are greater than one, (39) implies that if the two equilibria ever differ, then $\tilde{\pi}_t$ becomes unbounded, so that $\tilde{x}_t$ does as well. Because

x_t^* is bounded, $\hat{x}_t$ must eventually leave the interval $[\underline{x}, \bar{x}]$, which cannot happen in equilibrium. So we have a contradiction, and the first part of Proposition 6 is established.

Note that our construction implies that after any deviation in period t, the equilibrium outcomes from period $t+1$ are the desired outcomes. Thus, we have also established the second part of the proposition. QED

University of California, Los Angeles, Federal Reserve Bank of Minneapolis, and National Bureau of Economic Research
University of Minnesota and Federal Reserve Bank of Minneapolis
Federal Reserve Bank of Minneapolis, University of Minnesota, and National Bureau of Economic Research

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