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Rethinking Optimal Currency Areas

V. V. Chari

University of Minnesota
and Federal Reserve Bank of Minneapolis

Alessandro Dovis

Pennsylvania State University

Patrick J. Kehoe

University of Minnesota,
University College London,
and Federal Reserve Bank of Minneapolis

ABSTRACT

The classic optimal currency area criterion is that countries with more correlated shocks are better candidates to form a union. We show that when countries have credibility problems this simple criterion must be changed: Symmetric countries gain credibility when joining the union only when the shocks affecting credibility are not highly correlated. Our analysis provides a amended optimal currency area criterion that we argue is more relevant than the classic one. We illustrate our argument both for a reduced form model and for a relatively standard sticky-price general equilibrium model. We argue that our new criterion should lead to a rethinking of the massive amount of empirical work on optimal currency areas.
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The traditional case for flexible exchange rates and against a monetary union with a single currency dates back to at least Friedman (1953) and Mundell (1961). (See Dellas and Tavlas (2009) for a survey.) The argument is that with flexible exchange rates countries can tailor their monetary policy to respond to their idiosyncratic shocks while in a monetary union countries cannot. This inability to set monetary policy independently has been viewed as the major cost of a joining monetary union. Moreover, this cost is larger the greater is the variability of country-specific shocks. This traditional case implicitly assumes that countries had no credibility problems. Here we argue that when countries face substantial credibility problems, the loss of monetary independence can be a major benefit of joining a monetary union. Indeed, this benefit can increase with the variability of country-specific shocks and can lead a monetary union to be preferred to flexible exchange rates.

Some early work also considered credibility problems but with a very different institutional arrangement for how policies are set than the one considered here. For example, Friedman (1973) argued that for some countries it may be optimal to forswear a flexible exchange rate system and go beyond a monetary union all the way to "dollarization" in which a country simply abandons its currency. Specifically, Friedman argued that for a country with severe credibility problems it may be optimal to give up its currency and adopt the currency of another country, called an *anchor* currency.

> The surest way to avoid using inflation as a deliberate method of taxation is to unify the country's currency with the currency of some other country or countries. In this case, the country would not have any monetary policy of its own. It would, as it were, tie its monetary policy to the kite of the monetary policy of another country—preferably a more developed, larger, and relatively stable country.

This latter view of Friedman has been formalized by Alesina and Barro (2002). We interpret this work as making the case for *dollarization* for countries with credibility problems. A key institutional assumption of this work is that after dollarization the monetary policy of the country with the anchor currency is completely unaffected by the presence of another country that uses the same currency. This assumption seems particularly applicable when a small country, such as Ecuador, adopts the currency of a large country, such as the United

States, and forswears all explicit and implicit influence on the monetary policies followed by the large country. This assumption seems far less applicable to situations in which groups of countries come together in a monetary union and set up institutional arrangements in which they jointly decide on monetary policy.

In monetary unions, such as the European Monetary Union, monetary policy is made jointly by representatives of all countries in the union. A key distinction between our work on monetary unions and the existing work on dollarization is that in our work upon forming a union the members jointly decide on monetary policy in a way that takes account of the impact of policy on all members. Such a policy-making process raises the possibility that the union as a whole will suffer from the same type of credibility problems that the individual members face on their own. This possibility is especially acute if countries are symmetric with respect to their credibility problems. The question we address is how can symmetric countries increase their credibility by forming a union in which they jointly decide on monetary policy?

We analyze these issues in two models. The first is a reduced form model along the lines of Kydland and Prescott (1977) and Barro and Gordon (1983). The second is a simple sticky price model in the spirit of Gali and Monacelli (2005) and Farhi and Werning (2013).

Consider first the reduced form model. The monetary authority's objective is to minimize the deviations of unemployment from its natural level and the deviations of inflation from zero. There are two types of shocks to the natural level of unemployment: *ex-ante* shocks that are realized before price setters set their prices and *ex-post* shocks that are realized after. Both shocks have aggregate and idiosyncratic components.

Under commitment, the model is consistent with the standard Friedman–Mundell argument in favor of flexible exchange rates. With commitment the monetary authority finds it not optimal to respond to any ex-ante shocks. The reason is that if it does, the price setters will offset this response in their choice of prices and the net result will be no change in unemployment and an undesirable increase in the variability of inflation. The monetary authority, however, does find it optimal to respond to ex-post shocks since by doing so the authority can make the variability of unemployment lower. In a union the inability to respond to the idiosyncratic component of ex-post shocks raises the variability of unemployment and lowers welfare relative to a system of flexible exchange rates.

Our new result occurs when monetary authorities lack commitment. Here the role of ex ante shocks is critical. After the private agents set their prices, the monetary authority is tempted to engineer a surprise inflation that depends on the level of these shocks in order to reduce unemployment. In equilibrium, the private agents accurately forecast this policy and undo the effects of monetary policy on unemployment. Hence, the net effect of these forces is that, in equilibrium, the ex ante shocks leads the monetary authority to simply increase undesirable inflation variability.

In a union, in contrast, the monetary authority is unable to respond to the idiosyncratic component of the ex ante shocks. Hence, in equilibrium the variability of undesirable inflation is lower than under flexible exchange rates. Thus, in this model entering a monetary union is essentially a commitment device to less variable undesirable inflation that arises from reacting to ex ante shocks. This force tends to raise the value of the union relative to flexible exchange rates. Of course, in a union the monetary authority is also unable to respond to ex-post shocks, even though it is desirable to do so for the standard Friedman-Mundell reasons. This force tends to lower the value of the union relative to flexible exchange rates. Overall, if the variability of ex ante shocks is sufficiently large relative to that of ex post shocks then the credibility-enhancing benefits of the monetary union outweigh the standard Friedman–Mundell flexibility costs and the union is preferred to flexible exchange rates.

We then turn to a general equilibrium monetary model that is related to those of Obstfeld and Rogoff (1995), Gali and Monacelli (2005), and especially Farhi and Werning (2013). The economy consists of a continuum of ex-ante identical countries, each of which uses labor to produce traded and nontraded goods. The only shocks in the model are to the production of nontraded goods: the production function for each of the nontraded goods producers is subject to country-specific shocks and aggregate shocks to *productivity shocks* and to shocks to the the elasticity of substitution between the varieties of nontraded goods which we would refer to as *markup shocks*. To keep the analysis simple we purposefully abstract from the standard sources of gains from a monetary union, namely the reduction in transactions costs in trade. By doing so we highlight our main result: when countries have credibility problems, the inability of monetary policy to respond to idiosyncratic shocks in a monetary union may be a benefit rather than a cost.

The model features two key frictions. The first is that nontraded goods have sticky prices and are produced by monopolistically competitive firms. In each period, nontraded goods firms set their prices after the markup shocks are realized, but before either productivity shocks are realized or monetary policy has been set. These firms set their prices as a markup over their expected marginal costs and hence distort downward the production of nontraded goods. This distortion gives the monetary authority an incentive to engineer a surprise inflation so as to diminish the effective markup and increase the production of nontraded goods. This incentive is stronger the larger is the value of the markup shock. In contrast, the traded goods sector have flexible prices and are produced by competitive firms and hence have no such distortions.

The second friction is that purchases of traded goods must be made with money brought into the period while the purchase of nontraded goods are made with credit. This feature of the model generates costs for both surprise inflation and for expected inflation. (Other work that have used a similar device includes Svensson (1983), Nicolini (1998), and Albanesi, Chari, and Christiano (2003).) In the model a surprise inflation inefficiently lowers the consumption of traded goods ex post while an expected inflation distorts the consumption of the goods purchased with money–the traded goods–by raising the costs of purchasing them. (Notice that the presence of this second friction implies that in an equilibrium without commitment the monetary authority balances the benefits of surprise inflation against these costs and this friction leads to an interior solution for inflation.)

In our model, if the monetary authorities have no credibility issues then the standard Friedman-Mundell argument for flexible exchange rates holds. Specifically, when the monetary authority can commit to its policy, the flexible exchange rates regime is always preferable to the monetary union. Here membership in the monetary union simply restricts a policy instrument and adds an extra constraint to the Ramsey problem. This constraint binds whenever productivity shocks have an idiosyncratic component and leads welfare in the union to be lower then welfare under flexible exchange rates.

The economics behind the Friedman–Mundell effects is straightforward. When the idiosyncratic productivity of nontraded goods in a country is high, efficiency requires reducing the relative price of nontraded goods. Since nontraded goods prices are sticky, under flexible exchange rates this relative price reduction can be accomplished by an increase in the price of traded goods—a devaluation. In a monetary union no such devaluation can occur. Of course, it is possible to increase the price of all traded goods in the union by engineering a union-wide increase in the price of traded goods, but such an increase is not optimal in response to an idiosyncratic shock in one country. Hence, the monetary union restricts the ability of monetary policy to ensure efficient adjustment to idiosyncratic productivity shocks. The ex ante cost of this restriction is greater the larger is the volatility of idiosyncratic productivity shocks. In sum, under commitment our model is consistent with the standard argument for flexible exchange rates: flexible exchange rates helps to minimize the distortions imposed by sticky prices as suggested by Friedman (1953) and formalized by Gali and Monacelli (2005).

The more interesting analysis is what happens when countries have credibility issues. We model lack of commitment by considering a monetary authority that sets its policy in a Markovian fashion. Our novel result is that when countries have credibility problems, the standard Friedman–Mundell logic can be overturned. In particular, when the idiosyncratic component of the markup shocks is sufficiently high, countries can gain from giving up their monetary independence when moving from a system of flexible exchange rates to a monetary union. The key idea is that giving up the ability to target policy to country-specific markup shocks can raise credibility and hence raise welfare as long these credibility gains outweigh the standard Friedman–Mundell costs of being unable to target country-specific productivity shocks.

To understand the credibility gains in a union, consider what happens under flexible exchange rates after the realization of a high markup shock. Under flexible exchange rates this high shock increases the temptation of the monetary authority to generate a surprise inflation to reduce the monopoly distortion in the non-traded sector. In equilibrium this temptation is frustrated by the behavior of the sticky price firm: upon seeing a high markup shock, the nontraded goods firms anticipate the monetary authority's action and simply increase their price. By so doing these firms undo the real effects of the monetary policy. Hence, in equilibrium, the increase in the monetary authority's temptation due to the shock results only in a higher and more volatile inflation. Such inflation is welfare reducing because it generates a distortion in the tradable good sector: the high inflation increases the effective cost of traded goods and introduces a wedge between the marginal rate of substitution between labor and consumption of traded goods and the marginal rate of transformation between these same goods.

In contrast, in a monetary union, the union-wide monetary authority reacts only to union-wide variation in the markup shock. This inability to react to idiosyncratic markup shocks results in a lower volatility in the distortions in the traded good sector and thus, by itself, leads to higher welfare in the union. Of course, even here the inability to react to idiosyncratic productivity shocks, by itself, leads to lower welfare in the union. Overall, the Markov equilibrium in a monetary union has more volatile distortions in the non-traded sector and lower in the traded sector relative to the Markov equilibrium under flexible exchange rates. As long at the variability in idiosyncratic markup shocks is sufficiently large relative to the idiosyncratic volatility of productivity shock, the credibility gains from a monetary union outweigh the standard costs, and a monetary union is preferred to a system of flexible exchange rates.

Our model of a monetary union differs from some in the literature. We assume that countries that join the union stay in the union. In our setup as long as countries that join a union cannot leave the one until the end of the current period our analysis is unchanged. This assumption mimics that in Fuchs and Lippi (2006).

1. A Monetary Economy

Our monetary economy builds on the work of Obstfeld and Rogoff (1995), Gali and Monacelli (2005), and especially Farhi and Werning (2013).

The economy consists of a continuum of countries, each of which produces traded and nontraded goods and uses currency to purchase goods. The traded goods sector in each country is perfectly competitive. The nontraded goods consists of a continuum of firms each of which produces a differentiated product. The production function for each of the nontraded goods producers is subject to both aggregate and country-specific shocks to *productivity* and to the elasticity of substitution between the varieties of nontraded goods, referred to as *markup* shocks. The traded goods prices are flexible and they are bought with cash while the nontraded goods prices are sticky and they are bought with credit.

EXPLAIN role of ingredients....

We have purposefully chosen the ingredients of our model to be

We begin by describing the equilibrium for exogenous sequences of policies. We then turn to the classic comparison of a flexible exchange rate system and a currency union in an environment in which each monetary authority is fully committed to its policies. In the flexible exchange rate system the nominal price of traded goods can differ across countries while in a currency union this price must be equated across countries. We show that with commitment, the lack of monetary independence makes the currency union less desirable than a system of flexible exchange rates. We turn to making the same comparison when monetary authorities have no such commitment and instead set policy in a Markovian fashion. The inability to react to the idiosyncratic component of markup shocks leads the union to have credibility gains, while the inability to react to the idiosyncratic component of productivity shocks leads to Friedman–Mundell costs. Our main result is that if the variability of the idiosyncratic markup shocks is sufficiently high relative to that of the idiosyncratic productivity shocks then a monetary union is preferable to a regime of flexible exchange rates.

A. Environment

In each period t, an i.i.d. aggregate shock $z_t = (z_{1t}, z_{2t}) \in Z$ is drawn and each of a continuum of countries draws a vector of idiosyncratic shocks $v_t = (v_{1t}, v_{2t}) \in V$ which are i.i.d. both over time and across countries. The probability of aggregate shocks is $f(z_{1t}, z_{2t}) = f^1(z_{1t})f^2(z_{2t})$ and the probability of the idiosyncratic shocks is $g(v_{1t}, v_{2t}) = g^1(v_{1t})g^2(v_{2t})$. Here Z and V are finite sets. We let $s_t = (s_{1t}, s_{2t})$ with $s_{it} = (z_{it}, v_{it})$ and let $h(s_t) = h^1(s_{1t})h^2(s_{2t})$ with $h^i(s_{it}) = f^i(z_{it})g^i(s_{it})$. These aggregate and idiosyncratic shocks are to the nontraded goods sector and affect the elasticity of substitution between goods in this sector denoted $\theta(s_{1t})$ and referred to as markup shock and the productivity in this sector denoted $A(s_{2t})$. We let s^t and $h_t(s^t)$ denote the history and probability of these shocks and use similar notation for any components of these shocks.

The timing of events with a period is the following: first the markup shocks are realized, then the sticky price firms make their decisions, then the productivity shocks are realized, then the monetary authority chooses its policy, then households and flexible price firms make their decisions.

Production of Traded and Nontraded Goods

Consider first the production of traded and nontraded goods. The production function for a traded goods in a given country is simply $Y_{Tt}(s^t) = L_{Tt}(s^t)$ where $Y_{Tt}(s^t)$ is the output of traded goods and $L_{Tt}(s^t)$ are the inputs of labor in the traded goods sector. The problem of traded goods firms is then to solve

$$\max_{L_{Tt}(s^t)} P_{Tt}(s^t)L_{Tt}(s^t) - W_t(s^t)L_{Tt}(s^t)$$

so that in equilibrium

$$(1) \quad P_{Tt}(s^t) = W_t(s^t).$$

The non-traded good in any given country is produced by a competitive final consumption firm using $j \in [0,1]$ intermediates according to

$$(2) \quad Y_{Nt}(s^t) = \left[\int Y_N(j, s^t)^{\theta(s_{1t})} dj\right]^{1/\theta(s_{1t})}.$$

This firm maximizes

$$P_N(s^{t-1}, s_{1t})Y_{Nt}(s^t) - \int P_N(j, s^{t-1}, s_{1t})y_{Nt}(j, s^t)dj$$

where the notation makes clear that, consistent with our timing assumption, the prices of nontraded goods cannot vary with s_{2t}. The demand for an intermediate of type j is thus given by

$$(3) \quad Y_{Nt}(j, s^t) = \left(\frac{P_{Nt}(s^{t-1}, s_{1t})}{P_{Nt}(j, s^{t-1}, s_{1t})}\right)^{\frac{1}{1-\theta(s_{1t})}} Y_{Nt}(s^t)$$

The intermediate goods are produced by monopolistic competitive firms using a linear technology:

$$(4)\qquad Y_{Nt}(j,s^t)=A(s_{2t})L_N(j,s^t)$$

The problem of an intermediate good firm of type j is to choose $P=P_t(j,s^{t-1},s_{1t})$ to solve

$$(5)\qquad \max_P \sum_{s_t} Q_t(s^t)\left[P-W_t(s^t)\right]\left(\frac{P_{Nt}(s^t)}{P}\right)^{\frac{1}{1-\theta(s_{1t})}} Y_{Nt}(s^t)$$

subject to (4) where $Q_t(s^t)$ is the nominal stochastic discount factor. Throughout we will assume that $\theta(s_{1t})\in(0,1)$, so that the induced demands are elastic and that the optimal price for the monopolist is finite. The solution to this problem gives that for all intermediate goods producers j,

$$P_N(j,s^{t-1},s_{1t})=\frac{1}{\theta(s_{1t})}\frac{\sum_{s_{2t}} Q_t(s^t)C_{Nt}(s^t)\frac{W_t(s^t)}{A(s_t)}}{\sum_{s_{2t}} Q_t(s^t)C_{Nt}(s^t)}$$

where $1/\theta(s_{1t})$ is the markup in period t. Since this price does not depend on j we note that $P_{Nt}(j,s^{t-1},s_{1t})=P_{Nt}(s^{t-1},s_{1t})$. This result implies that the labor hired by each intermediate goods firm is the same so that $L_N(j,s^t)$ can be written $L_{Nt}(s^t)$ and the final output of nontraded goods is simply

$$(6)\qquad Y_{Nt}(s^t)=A(s_{2t})L_{Nt}(s^t).$$

Consumers and the Government

The consumers in any given country have preferences given by

$$(7)\qquad \sum_{t=0}^{\infty}\sum_{s^t}\beta^t h_t(s^t)U\left(C_{Tt}(s^t),C_{Nt}(s^t),L_t(s^t)\right)$$

where $C_{Tt}(s^t)$ is the consumption of traded goods, $C_{Nt}(s^t)$ is the consumption of the (final) nontraded good, and $L_t(s^t)$ is (total) labor supply. In most of our analysis we will specialize preference to be

$$(8) \quad U(C_T, C_N, L) = \alpha \log C_T + (1-\alpha)\log C_N - bL$$

The critical feature is quasi-linearity in labor which allows to obtain nice aggregation results along the lines of Lagos and Wright (200*).

Consumers are subject to a cash-in-advance constraint that requires them to buy traded goods at t using domestic money brought in from period $t-1$, namely $M_{t-1}(s^{t-1})$, so that

$$(9) \quad P_{Tt}(s^t)C_{Tt}(s^t) \leq M_{t-1}(s^{t-1}).$$

The budget constraint of the consumer is given by

$$(10) \quad \begin{aligned} & P_{Tt}(s^t)C_{Tt}(s^t) + P_N(s^{t-1}, s_{1t})C_{Nt}(s^t) + M_t(s^t) + B_t(s^t) \\ \leq\ & W_t(s^t)L_t(s^t) + M_{t-1}(s^{t-1}) + (1 + r_t(s^t))B_{t-1}(s^{t-1}) + T_t(s^t) + \Pi_t(s^t). \end{aligned}$$

where $T_t(s^t)$ are nominal transfers, $\Pi_t(s^t)$ are the profits from the nontraded goods firms, $r_t(s^t)$ is the nominal interest rate in the domestic currency, and $B_t(s^t)$ are nominal bonds.

Under (8) and our shock structure countries have no incentive to borrow and lend to each other so that in equilibrium

$$(11) \quad B_t(s^t) = 0$$

The first order conditions for the consumer are summarized by

$$(12) \quad \frac{U_{Nt}(s^t)}{P_{Nt}(s^{t-1}, s_{1t})} = -\frac{U_{Lt}(s^t)}{W_t(s^t)}$$

$$(13) \quad \frac{U_{Tt}(s^t)}{P_{Tt}(s^t)} = -\frac{U_{Lt}(s^t)}{W_t(s^t)} + \xi_t(s^t) \geq -\frac{U_{Lt}(s^t)}{W_t(s^t)}$$

$$(14) \quad \frac{U_{Nt}(s^t)}{P_{Nt}(s^{t-1})} = \beta \sum_{s^{t+1}} h(s^{t+1}|s^t)\frac{U_T(s^{t+1})}{P_{Tt+1}(s^{t+1})}$$

$$(15)\quad \frac{1}{1+r_t(s^t)} = \beta \sum_{s^{t+1}} h_{t+1}(s^{t+1}|s^t) \frac{U_N(s^{t+1})}{P_{Nt}(s^t, s_{1t+1})} \frac{P_{Nt}(s^{t-1}, s_{1t})}{U_{Nt}(s^t)}.$$

where $\xi_t(s^t) \geq 0$ is the (normalized) multiplier on the cash-in-advance constraint. Notice also that the nominal stochastic discount factor for the country is

$$(16)\quad Q_{t+1}(s^{t+1}) = \beta h_{t+1}(s^{t+1}|s^t) \frac{U_N(s^{t+1})}{P_{Nt}(s^t, s_{1t+1})} \frac{P_{Nt}(s^{t-1}, s_{1t})}{U_{Nt}(s^t)}$$

where $Q_t(s^t)$ is the price of a state-contingent claim to local currency units at s^t in units of local currency at s^t. This is the relevant price that firms use to discount profits in (5).[1] The monetary authority's budget constraint is simply that newly created money is transferred to consumers in a lump-sum fashion

$$(17)\quad T_t(s^t) = M_t(s^t) - M_{t-1}(s^{t-1}),$$

In this economy policies can be be described as a sequence of interest rates, money supplies, and transfers that satisfy (15) and (17). In terms of what follows, we can either let the monetary authority a nominal interest rate policy and letting nominal transfers and money growth being endogenously determined or we can let the monetary authority choose money growth rates and letting interest rates and transfers be endogenously determined.

We focus on symmetric equilibria, in which any two countries with the same history of idiosyncratic shocks s^t have the same allocations, policies, and prices. Given this symmetry, consider first the definition of an equilibrium with flexible exchange rates. Given initial conditions $\{M_{-1}, B_{-1}\}$ and a policy $\{r_t(s^t), M_t(s^t), T_t(s^t)\}$ an *equilibrium with flexible exchange rates* is a set of allocations $X = \{C_{Tt}(s^t), C_{Nt}(s^t), Y_{Tt}(s^t), Y_{Nt}(s^t), L_{Tt}(s^t), L_{Nt}(s^t), L_t(s^t), M_{t-1}(s^{t-1})\}$ and price system $P = \{W_t(s^t), P_{Tt}(s^t), P_{Nt}(s^{t-1}, s_{1t}), Q_t(s^t), r_t(s^t)\}$ such that: i) at these prices, the decisions of households are optimal, ii) at these prices, the decisions of firms are

[1] To check this claim add to the left side of the consumer's budget constraint period t purchases of nominal contigent claims $\sum_{s^{t+1}} Q_{t+1}(s^{t+1}) D_{t+1}(s^{t+1})$ and to the right side the payments for period $t-1$ purchases $D_t(s^t)$ and note that the resulting first order condition gives the formula for $Q_{t+1}(s^{t+1})$.

optimal, iii) the labor market clears in each country

$$(18)\quad L_{Nt}(s^t) + L_{Tt}(s^t) = L_t(s^t),$$

iv) the traded and nontraded goods markets clear

$$(19)\quad C_{Tt}(s^t) = Y_{Tt}(s^t), C_{Nt}(s^t) = Y_{Nt}(s^t),$$

v) the monetary authority's budget constraint holds and the interest rate $r_t(s^t)$ satisfies (15) and (17).

So far we have expressed each country's prices in units of its own currency. Since the law of one price holds for traded goods, we can write the nominal exchange rate between a particular country and all others as

$$(20)\quad E(s^t) = \frac{P_{Tt}(s^t)}{\int_{v^t} P_{Tt}(z^t, v^t) g^t(v^t) dv^t}$$

where $g^t(v^t) = g(v_0) \ldots g(v_t)$ and the term on the bottom is the simple average over all countries, where countries are identified by the history of their idiosyncratic shocks v^t, where we have used the law of large numbers to avoid recording the realization of idiosyncratic shocks for all countries besides the particular country of interest.

We model a monetary union as the restriction that the nominal price of traded goods is the same for all countries, so that at time t, if one country has a history $s^t = (z^t, v^t)$ and another has history $\tilde{s}^t = (z^t, \tilde{v}^t)$ then $P_{Tt}(s^t) = P_{Tt}(\tilde{s}^t)$. Hence, P_{Tt} depends on the history of aggregate shocks but not on the history of any country's idiosyncratic shocks. An *equilibrium with fixed exchange rates* is defined analogously to an equilibrium with flexible exchange rates with the added restriction that $E(s^t) = 1$ for all s^t which, using (20) implies that for any s^t and $\tilde{s}^t$,

$$(21)\quad P_{Tt}(s^t) = P_{Tt}(\tilde{s}^t)$$

for all $s^t = (z^t, v^t)$ and $\tilde{s}^t = (z^t, \tilde{v}^t)$. Note that here we model the union as restricting the nominal exchange rate between countries to be equal to 1, but otherwise we let the rest of monetary policy differ across countries.

2. Optimal Policy with Commitment

We turn now to analyzing optimal policy under flexible exchange rates and in a monetary union. We will show that the lack of monetary independence in a monetary union imposes a loss on member countries. The intuition for this result is based on the standard Friedman-Mundell logic: under fixed exchange rates countries are less able to target monetary policy to their country specific shocks. Of course, since we have abstracted from the standard Mundellian gains to trade that accompanies a monetary union this result is consistent with Mundell's optimal currency criterion. For any given gains from trade of a currency union (here zero) countries should join the union only if the idiosyncratic component of their shocks is small enough.

We start by defining the *Ramsey problem* for a country under *flexible exchange rates.* The problem is to choose allocations X and a price system P given initial conditions $\{M_{-1}, B_{-1}\}$ to maximize consumer utility (7) subject to the consumer and firm first order conditions and the resource constraints.

In a monetary union the price for traded goods cannot vary with idiosyncratic shocks, in that (21) holds. The *Ramsey problem* in a *monetary union* can thus be written as choosing allocations X and a price system P given initial conditions $\{M_{-1}, B_{-1}\}$ to maximize consumer utility (7) subject to the consumer and firm first order conditions and the resource constraints and the additional constraint (21) for all $s^t = (z^t, v^t)$ and $\tilde{s}^t = (z^t, \tilde{v}^t)$.

The fact that the Ramsey problem under flexible exchange rates is a more relaxed version of the Ramsey problem in a monetary union immediately implies the following result:

Proposition 1. The Ramsey problem under flexible exchange rates leads to higher welfare than the Ramsey problem in a monetary union and strictly higher welfare as long as the variance of idiosyncratic productivity shocks is strictly positive.

PROOF IS WRONG

conjecture:

if

$$U = U_1(C_N) + U_2(C_T) + V(L)$$

and U_1 is $C_N^{1+\alpha}$

write of foc of Ramsey and show that under these conditions that flexible price equilibrium has

Proof. We begin by establishing an upper bound for welfare. To do so consider a flexible price version of our economy without any money, in the sense that the cash in advance constraints are dropped. The first order conditions for the consumer and the firm imply

$$(22) \quad \frac{U_{Nt}(s^t)}{U_{Tt}(s^t)} = \frac{1}{\theta(s^{t-1}, s_{1t})A(s^t)}$$

$$(23) \quad \frac{U_{Nt}(s^t)}{P_{Nt}(s^t)} = -\frac{U_{Lt}(s^t)}{W_t(s^t)}$$

and $W_t(s^t) = P_{Tt}(s^t)$. Clearly, up to the monopoly markup, (22) implies that the marginal rate of substitution between nontraded and traded goods equals the marginal rate of transformation.

These allocations can be implemented as an equilibrium in our economy (with money and sticky prices) under flexible exchange rates. To do so let monetary policy follow the Friedman rule of setting nominal interest rates to zero so that the cash-in-advance constraint is slack. Then from the consumer's first order conditions for nontraded and traded goods (12) and (13) it follows that

$$(24) \quad \frac{U_{Nt}(s^t)}{U_{Tt}(s^t)} = \frac{P_{Nt}(s^{t-1}, s_{1t})}{P_{Tt}(s^t)}$$

Let $P_{Tt}(s^t) = P_{Nt}(s^{t-1}, s_{1t})\theta(s^{t-1}, s_{1t})A(s^t)$. It follows immediately that the marginal rate of substitution between nontraded and traded goods equals the marginal rate of transformation up to the monopoly markup in that (22) holds. Clearly, to satisfy profit maximization by traded goods producers we need $W_t(s^t) = P_{Tt}(s^t)$. To see that these choices are consistent with profit maximization by nontraded goods producers we note that at these choices the the sticky price first order condition (**??**) holds. Hence, we have shown the Ramsey problem under flexible exchange rates attains the upper bound for welfare.

We now show that the Ramsey problem under in a monetary union does not attain the upper bound. In a monetary union by following the Friedman rule the allocations satisfy (24). But, since the price of traded goods $P_T(s^t)$ cannot depend on the current period idiosyncratic shock v_{2t}, the relative price of nontraded to traded good $P_{Nt}(s^{t-1}, s_{1t})/P_{Tt}(s^t)$ cannot equal the markup-distorted marginal rate of transformation $\theta(s^{t-1}, s_{1t})/A(s^t)$ for all values of the idiosyncratic shock. Thus, the Ramsey problem in a monetary union does not attain the upper bound for welfare. *Q.E.D.*

For future use it is useful to characterize the Ramsey equilibrium when productivity is constant at A in all countries. In this case the cash-in-advance constraint is slack so that the Ramsey allocations satisfy

$$(25)\quad \frac{U_N(s^t)}{U_T(s^t)} = \frac{P_N(s^{t-1}, s_{1t})}{P_T(s^t)} = \frac{1}{\theta(s^{t-1}, s_{1t})A}$$

where the first equality follows from the consumer's first order conditions and the second from the firm's first order conditions. Clearly, the consumer's marginal rate of substitution between nontraded and traded goods equals their relative price. But, since the markup $\theta(s^{t-1}, s_{1t}) > 1$, this relative price is greater than the marginal rate of transformation between these goods because of the monopoly distortion. Intuitively, the monopolists set the price of nontraded goods too high so that the consumption of nontraded goods is too low relative to the socially efficient level.

PUT IN AGGREGATE SHOCKS:

We supplement this general intuition by working out the allocations and prices in closed form for our preference specification (8) in the case of no aggregate shocks Specifically, as we show in Appendix B, in this case under flexible exchange rates the consumption of nontraded goods is given by

$$(26)\quad C_N^R(v) = \frac{(1-\alpha)}{b}\theta(v_1)A(v_2),$$

the consumption of traded goods is given by $C_T = \alpha/b$, and labor is given by

$$(27)\quad L^R(v_1) = \frac{1}{b}\left[\alpha + (1-\alpha)\theta(v_1)\right]$$

In a monetary union in this case the consumption of nontraded goods is given by

$$(28)\quad C_N^{R,U}(v_1) = \frac{1-\alpha}{b}\frac{1}{\theta(v_1)}\frac{1}{\sum_{v_2} g^2(v_2)/A(v_2)},$$

the consumption of traded goods is $C_T = \alpha/b$, and labor is given by

$$(29)\quad L^{R,U}(v_1) = \frac{1}{b}\left[\alpha + (1-\alpha)\frac{\theta(v_1)/A(v_2)}{\sum_{\tilde{v}_2} g^2(\tilde{v}_2)/A(\tilde{v}_2)}\right]$$

Since the consumption of traded goods is equal across regimes and the expected value of labor supply is equal across regimes then difference in utility in the regimes is that due to the differences in the consumption of nontraded goods. That is,

$$EU^R - EU^{R,U} = (1-\alpha)\sum_{v_1,v_2} g^1(v_1)g^2(v_2)\left[\log C_N^R(v) - \log C_N^{R,U}(v)\right]$$

Proposition 2.Under (8) we have that the utility difference between the two regimes is given by

$$(30)\quad (1-\alpha)\sum_{z_2} f^2(z_2)\left[\log\left(\sum_{v_2} g^2(v_2)\frac{1}{A(v_2,z_2)}\right) - \sum g^2(v_2)\log\frac{1}{A(v_2,z_2)}\right] > 0$$

which equals $(1-\alpha)E_z\left[\log E_v\left(\frac{1}{A}|z\right) - E_v\left(\log\frac{1}{A}|z\right)\right]$.

Clearly, (30) is strictly positive since the log function is a concave function. We find it useful to consider the simple case in which $A(v_2,z_2) = A_v(v_2)A_z(z_2)$ and $A_v(v_2)$ is log normal with mean μ_v and variance σ_v^2. Here (30) reduces to $(1-\alpha)\sigma_v^2/2$ so that the losses in joining the union are increasing in the volatility of the idiosyncratic productivity shocks. [2]

[2] If A is log normal μ,σ then $A = e^{\mu+\sigma Z}$ where Z is normal 0,1 and $EA = e^{\mu+\sigma^2/2}$ Then $1/A = e^{-\mu-\sigma Z}$ and $E(1/A) = e^{-\mu+\sigma^2/2}$ so $\log E(1/A) = -\mu+\sigma^2/2$. Now $\log(1/A)$ is $\log e^{-\mu-\sigma Z} = -\mu-\sigma Z$ and $E(-\mu-\sigma Z) =$

In sum, we have shown that under commitment the greater the idiosyncratic variability of producitivity shocks, the greater the costs of joining a union. Moreover, markup shocks affect welfare the same way in both regimes and hence play no role in determining the costs of joining a union.

3. Optimal Policy without Commitment

Consider now the same physical environment except that the monetary authorities cannot commit. We model this lack of commitment as having these authorities as choose policies in a Markovian fashion.

Recall that in the environment with commitment, markup shocks play no role in determining the costs or benefits of joining a monetary union. We will show that in the environment without commitment, the markup shocks play a critical role in determining these costs and benefits. In particular, the more variable are markup shocks the larger are the gains from joining a monetary union. It turns out that the productivity shocks play similar roles with and without commitment. To focus on the role of markup shocks we assume for most of what follows that productivity is constant across countries and time. For ease of notation we drop the subscripts denoting first stage shocks and write (z_1, v_1) as (z, v) and drop the second stage shocks altogether.

The timing is the same as before. The first stage—the *sticky price* stage—occurs at the beginning of the period after the markup shocks associated with (z, v) have been realized. At this stage the sticky price firms make their decisions. At the next stage—the *policy* stage—monetary policy is set. Then at the *household* stage, the household and the flexible price firms make their decisions.

A. Flexible Exchange Rates

Under flexible exchange rates the allocations and prices in any given country are not affected by decision of households, firms, and governments in other countries. This

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$-\mu$ so

$$\log E(1/A) - E\log(1/A) = -\mu + \sigma^2/2 - (-\mu) = \sigma^2/2$$

observation allows us to focus on a particular country in isolation. We begin by describing the state variables for the sticky and flexible price firms, the households, and the monetary authority. We normalize all nominal variables by the beginning of period aggregate stock of money M_{-1} in the given country. With this normalization the normalized aggregate money stock is 1.

The *sticky price firm state* is the beginning of period exogenous shocks s_1. Let $p_N = P_N/\bar{M}_{-1}$ denote the normalized nontraded goods price and let $\bar{p}_N(s)$ denote the sticky price firm's normalized decision rule. The *monetary authority state* is $x_G = (p_N, s)$. Denote the growth rate of money by μ and let $\mu(x_G)$ denote the monetary authority's policy decision for this growth rate and use similar notation for transfers and nominal interest rates. Finally, the *household state* (m_H, x_H) consists of the normalized level of that household's money $m_H = M_H/M_{-1}$ and an aggregate component $x_H = (p_N, s, \mu)$. Denote the household decision rule for the consumption of the traded good C_T as $C_T(m_H, x_H)$ and use similar notation for other household choices. The *flexible price firm state* is x_H. Denote the rule for normalized traded goods prices as $\bar{p}_T(x_H)$.

With this notation in hand we can set up the consumer's problem as follows.

$$(31)\quad V(m_H, x_H) = \max_{C_T, C_N, L, m'_H} U(C_T, C_N, L) + \beta \sum_s h(s')V(m'_H, x'_H)$$

subject to the cash-in-advance constraint

$$\bar{p}_T(x_H)C_T \leq m_H$$

and the budget constraint

$$\bar{p}_T(x_H)C_T + p_N C_N + \mu m'_H \leq m_H + \bar{w}(x_H)L + [\mu(x_G) - 1] + \pi(x_H)$$

where m_H, p_N and μ are in the state. The new normalized price of nontraded goods

$$(32)\quad p'_N = \bar{p}_N(s').$$

The problem for the monetary authority can be written as choosing μ so that

$$(33)\quad W(x_G) = \max_{\mu} V(1, (x_G, \mu))$$

A *Markov equilibrium under flexible exchange rates* consists of sticky price decision rules $\bar{p}_N(s)$, monetary authority's policy decision $\mu(x_G)$,and value function $W(x_G)$, households decision rules $C_N(m_H, x_H)$, $C_T(m_H, x_H)$, $L(m_H, x_H)$, $m'_H(m_H, x_H)$, and value function $V(m_H, x_H)$, price rules $\bar{w}(x_H)$ and $\bar{p}_T(x_H)$, profit rules $\pi(x_H)$, such that i) the flexible price firm and the household decision rules are optimal in that the flexible price firms' price rule satisfies

$$(34)\quad \bar{p}_T(x_H) = \bar{w}(x_H)$$

and the household decision rules are optimal for problem (31) and the value function V and the profit rule satisfies

$$(35)\quad \pi(x_H) = \left(p_N - \frac{\bar{w}(x_H)}{A}\right) C_N(m, x_H)$$

ii) the sticky price firms' price rule satisfies

$$(36)\quad \bar{p}_N(s) = \frac{1}{\theta(s)} \frac{\bar{w}(x_H)}{A}$$

where x_H is induced by the sticky price firm's pricing rule and the monetary policy rule $\bar{\mu}$, iii) the market clearing conditions hold, $C_N(1, x_H) = AL_N(x_H)$, $C_T(1, x_H) = L_T(x_H)$, $L(1, x_H) = L_N(x_H) + L_T(x_H)$, as well as money market clearing in the current and future periods

$$(37)\quad m'_H(1, x_H) = 1$$

where x_H is induced from x_G by $\bar{\mu}(x_G)$, iv) government decision rule and value function solve (33)

To characterize this equilibrium, consider the problem faced by a monetary authority in choosing its policy. We find it convenient to write the problem in primal form in the sense that we think of this authority as directly choosing prices and allocations subject to the first order conditions of firms and consumers and the resource constraints. Under flexible exchange rates, the policy rule $\bar{\mu}(x_G)$ is part of a Markov equilibrium if solves the *primal Markov* problem

$$(38)\quad W^{flexible}(x_G) = \max_{p_T, C_T, C_N, L, \mu} U\left(C_T, C_N, L\right) + \beta \sum_{s} h(s') W^{flexible}\left(x'_G\right)$$

subject to

$$(39)\quad L = C_T + \frac{C_N}{A}.$$

$$(40)\quad \frac{U_N}{p_N} = -\frac{U_L}{p_T}$$

$$(41)\quad \frac{U_T}{p_T} \geq -\frac{U_L}{p_T}$$

$$(42)\quad p_T C_T \leq 1$$

where if (**??**) is a strict inequality then (**??**) holds as an equality, and

$$(43)\quad -\mu\frac{U_L}{p_T} = \beta \sum_{s'} h(s') \frac{U_T(1, x'_H)}{p_T(x'_H)}$$

Next we show that the monetary authority's problem is static. To see this result note that x'_G and x'_H are determined by future decision makers and are independent of the current money growth rate choice. Therefore, the continuation value $W\left(x'_G\right)$ and the right side of (43) are also independent of the current money growth rate choice. Since μ appears only in (43) we can drop μ and this constraint as a choice variable and solve the problem of maximizing current period utility $U\left(C_T, C_N, L\right)$ subject to (39)-(42).

Notice that these four constraints give 3 equations, since either the cash-in-advance constraint (42) is slack and (41) holds with equality or the cash in advance constraint binds and (41) is slack. Hence, we can think of this problem as determining the *best response of the monetary authority* $p^f_T(p_N, s)$ to a given choice of nontraded goods price by the nontraded goods firms and the other three constraints as determining C_T, C_N,and L as a function of $p_T^f(p_N, s)$.

To sharpen our characterization of this best response let (C_T^*, C_N^*, L^*) denote the solution to this problem dropping the cash in advance constraint. Note the resulting allocations are efficient in that they maximize utility subject to the resource constraint. Clearly, the solution is to set $p_T = p_N A$. Thus, if at these prices the cash in advance constraint is satisfied, in that $p_N A C_T^* \leq 1$, then this allocation solves the monetary authority's problem. If $p_N A C_T^* > 1$ then the cash-in-advance constraint binds in the monetary authority's problem and the resulting consumption of traded goods is distorted downward relative to the efficient allocations in that $C_T < C_T^*$.

We summarize this discussion in the following lemma.

Lemma 1. If $p_N > 1/AC_T^*$ then the monetary authority's problem (38) has a binding cash-in-advance constraint binds and the resulting allocation of traded goods $C_T < C_T^*$. If $p_N \leq 1/AC_T^*$ then the cash-in-advance constraint is slack and $p_T = p_N A$.

Consider now the problem of the sticky price producers. Each of these producers takes as given the representative nontraded goods price p_N and predicts that the price of traded goods will be given by $\bar{p}_T(p_N, s)$. Using (34) each nontraded goods producer predicts that the wage rate will also be given by $\bar{p}_T(p_N, s)$. Thus, each individual nontraded goods producer sets their prices of nontraded goods at a markup over the price of predicted traded goods. Hence, in any equilibrium, the representative price of traded goods at state s must be a fixed point of

$$(44) \quad \bar{p}_T(s) = p_T^{flex}(\frac{\bar{p}_T(s)}{\theta(s)A}, s)$$

Once we have the fixed point $\bar{p}_T(s)$, the rest of the allocations are determined by the constraints on the monetary authority's problem.

We now argue that the cash in advance constraint is binding in any Markov equilibrium. To do so suppose by way of contradiction that the cash-in-advance constraint is slack. Lemma 1 then implies that $p_T = p_N A$. But since $\theta(s) < 1$ for all s, any such solution will not satisfy the fixed point property (44). Thus, the cash in advance constraint must be binding and the consumption of traded goods is distorted downwards relative to the efficient outcomes.

To understand the time inconsistency problem, consider starting at the Ramsey allocations that satisfy (25) in which the relative price of nontraded goods is higher than the marginal rate of transformation between nontraded and traded goods. The monetary authority has an incentive to move the marginal rate of substitution closer to the marginal rate of transformation between these goods. Given that the price of nontraded goods is fixed the monetary authority can do so by raising the price of traded goods and the wage rate. This policy can interpreted as devaluing the currency. In equilibrium the cash-in-advance constraint binds and this policy is costly because raising the price of traded goods reduces the consumption of traded goods. The best response of the monetary authority balances these costs and benefits.

With our functional form (8) using the result that the cash in advance constraint must be binding we have that the primal Markov problem can be written as, given $p_N = p_N(s)$ and $\theta = \theta(s)$, choose (C_N, C_T, p_T) to solve

$$\max \alpha \log C_T + (1-\alpha)\log C_N - b\left[C_T + C_N/A\right] \tag{45}$$

subject to

$$C_T = \frac{1}{p_T}$$

$$C_N = \frac{1-\alpha}{b}\frac{p_T}{p_N(s)}$$

with associated expected utility The best response of the monetary authority can be written as

$$p_T = F(\frac{1}{p_N}) \tag{46}$$

for a quadratic function F defined in the Appendix. Substituting that wages equal the price of the traded goods, the price chosen by the monopolist is

$$p_N = \frac{1}{\theta}\frac{p_T}{A}.$$

Thus, we can summarize the equilibrium $p_T(s)$ as defined by $p_T(s) = F(\theta(s)A/p_T(s))$. Solving this fixed point problem gives the equilibrium $p_T(s)$ and is associated with the Markov allocations that we summarize in the next lemma. In order to ensure the existence of an equilibrium we impose

$$(47)\quad \theta(s) > \frac{1-2\alpha}{1-\alpha} \text{ for all } s$$

This assumption guarantees that the monopoly distortions are sufficiently small so that a Markov equilibrium exists. At an intuitive level, if $\theta(s) < (1-2\alpha)/(1-\alpha)$ then the gains to the government of inflating in order to reduce the distortion ex post are sufficiently high that no matter what the level of p_N the government will always have an incentive to raise the price of traded goods so that no equilibrium exists.

Lemma 2. Under (8) and (47), the allocations in the Markov equilibrium with flexible exchange rates is given by

$$(48)\quad C_T(s) = \frac{\alpha}{b} - \frac{1-\alpha}{b}(1-\theta(s))$$

$$(49)\quad C_N(s) = \frac{1-\alpha}{b}\theta(s)A$$

and $L(s) = C_T(s) + C_N(s)/A$.

In Figure 1 we graph the best response of the monetary authority and the price chosen for nontraded goods for $\theta \in \{\theta_L, \theta_H\}$ with $\theta_L < \theta_H$. Under θ_L there is a lower markup than under θ_H since $1/\theta_L > 1/\theta_H$. As the graph shows, when markups are high the consumption of traded goods is low. Under flexible exchange rates a mean preserving spread in θ leads to a mean preserving spread in both the consumption of traded and nontraded goods and hence lowers welfare.

B. Monetary Union

We begin by describing the state variables for the sticky and flexible price firms, the households, and the monetary authority. We normalize all nominal variables by the beginning of period aggregate stock of money $\bar{M}_{-1}$. Note that for an arbitrary measure Λ over entering nominal money stocks over countries we can define the aggregate nominal money stock as

$$\bar{M}_{-1} = \int M_{-1} d\Lambda(M_{-1})$$

The *sticky price firm state* in (x_F, S_F) where $x_F = (m, v)$ and $S_F = (z, \lambda_F)$ and λ_F is a measure over x_F. Denote the sticky price firm's normalized decision rule as $\bar{p}_N(x_F, S_F)$.

The *monetary authority state* in a country consists of a country-specific component $x_G = (m, p_N, v)$ and an aggregate component $S_G = (z, \lambda_G)$ where λ_G is a measure over x_G. The corresponding union-wide monetary authority state is simply S_G. Denote the monetary authority's policy decision for money as $\mu(x_G, S_G)$ and use similar notation for transfers and nominal interest rates.

Finally, the *household state* has a household-specific component, a country-specific component, and a aggregate component. The household-specific component is the normalized level of that household's money $m_H = M_H/\bar{M}_{-1}$. The country-specific component $x_H = (m, p_N, v, \mu)$ consists of the normalized money balances for the country as a whole $m = M/\bar{M}_{-1}$, the normalized price of nontraded goods $p_N = P_N/\bar{M}_{-1}$, the idiosyncratic shocks v, and the country-specific growth rate of money μ. The aggregate component $S_H = (z, \lambda_H)$ consists of the aggregate shock z and a measure λ_H over the country-specific components (m, p_N, v, μ) for all countries. Thus, the household state is (m_H, x_H, S_H). Denote the household decision rule for the consumption of the traded good C_T as $C_T(m_H, x_H, S_H)$ and use similar notation for other household choices. The *flexible price firm state* is (x_H, S_H). Denote the rule for normalized traded goods prices as $\bar{p}_T(x_H, S_H)$. Note for later use that the marginal measure of λ_H over x_G is λ_G and the marginal measure of either λ_H or λ_G over x_F is λ_F. We will use these properties repeatedly below.

A formal definition of a Markov equilibrium is relegated to the appendix. Consider the problem faced by a monetary authority in choosing its policy. As in the case of flexible exchange rates, we find it convenient to write the problem in primal form in the sense that we think of this authority as directly choosing prices and allocations for all countries in the union subject to the first order conditions of firms and consumers and the resource constraints. This authority takes as given the current state S_G and the future monetary policy rule $\bar{\mu}(x_G, S_G)$. Since countries are indexed by x_G, we think of the government as choosing say $C_T(x_G)$ and we use similar notation for other variables. Of course, in a union the price of traded goods does not vary with the country so we think of the monetary authority in each aggregate state as choosing a common price p_T for all countries. The optimal rules, of course, also depend on the aggregate state and can be written $\bar{C}_T(x_G, S_G)$ and so on. To make it clear the nontraded goods price p_N for some particular country is part of that country's state $x_G = (m, p_N, v)$ we write $p_N(x_G)$

We then have that the policy rule $\bar{\mu}(x_G, S_G)$ is part of a Markov equilibrium if solves the functional equation

(50)

$$W^{union}(S_G) = \max_{p_T, C_T(x_G), C_N(x_G), L(x_G), \mu(x_G)} \int U\left(C_T(x_G), C_N(x_G), L(x_G)\right) d\lambda_G + \beta \sum_s h(s') W^{union}\left(x'_G, S'_G\right)$$

subject to, for each country with $x_G = (m, p_N, v)$

(51) $$\frac{U_N(x_G)}{p_N(x_G)} = -\frac{U_L(x_G)}{p_T}$$

(52) $$\frac{U_T(x_G)}{p_T} \geq -\frac{U_L(x_G)}{p_T}$$

(53) $$p_T C_T(x_G) \leq m$$

where if (53) is a strict inequality then (52) holds as an equality, and

(54) $$\gamma \frac{-U_L(x_G)}{p_T} = \beta \sum_{s'} h(s') \frac{U_T(m'_H, x'_H, S'_H)}{p_T(x'_H, S'_H)}$$

(55) $$L(x_G) = C_T\left(x_G\right) + \frac{C_N(x_G)}{A}$$

where the evolution of the individual state $x'_G = (m', p'_N, v')$ is given by $\mu m/\gamma$ where $\gamma = \int [\bar{\mu}(x_G, S_G)m]\, d\lambda_G(x_G)$, $p'_N = \bar{p}_N(m', \nu_1, S'_F)$ and the evolution of the aggregate state for the firms S'_F and the aggregate state for the government S'_G is determined by the decision rules by households, firms.

We turn now to showing that under our preference specification we can restrict attention to Markov equilibria with degenerate distributions for money holdings.

Lemma 3. Under the preference specification (8) in any Markov equilibrium in a monetary union, given any initial distribution of money at the beginning of the period then end of period money holdings are concentrated on a single point.

From now on we will choose the date 0 initial nominal money holdings of all countries to be equal and then use Lemma 3 to guarantee that these money holdings will continue to be equal over time.

Note that the proof of this lemma has two ideas. The first is that if two agents have differing money holdings, say $m_1 < m_2$,.and the cash-in-advance constraint binds in at least one state in the next period then these agents experience different consumption levels of traded goods in at least one state, and hence differing levels of expected marginal utility of traded goods consumption. But, for each of these agents the first order conditions imply that the marginal disutility of labor must be equated to the expected marginal utility of traded goods consumption. Since under (8) the marginal disutility of labor of each agent is equal so must be the expected marginal utility of traded goods consumption, which is a contradiction.

The second idea is that in equilibrium if the markup is positive in some state the monetary authority will continue to increase the price of traded goods until the benefit of surprise inflation is balanced against some cost from this inflation. Since the cost from surprise inflation only occurs when the cash-in-advance constraint is binding, then we know that the cash-in-advance constraint must bind whenever the markup is positive. Combining these two ideas gives Lemma 3.

It follows from Lemma 3 that if we choose the initial distribution of money holdings across countries is degenerate then in each future period it will continue to be degenerate. Thus, the normalized level of money balances m_H is one in each country in all periods. (Of course, the absolute level of money balances will typically be changing over time.) Thus, as in the flexible exchange rate case, we can drop m from the individual state, and thus λ_G is a distribution only over (p_N, v). Since the price of nontraded goods can depend only on (z, v), the distribution λ_G over $(p_N(z, v), v)$ is equivalent to the distribution $g(v)$ over v.

Also, Lemma 3 implies that, as in the flexible exchange rate case, both the continuation utility and future allocations are unaffected by current decisions. Thus, the problem of the monetary union reduces to the static problem of maximizing current utility. Morever, since the right side of (54) does not depend on current choices, we can let the first order condition (54) define γ and drop this condition.

Here, the beginning of period aggregate state is a distribution of nontraded good prices $\{p_N(s)\}$ and an aggregate shock z. Given this state, the price of traded goods and the allocations solve

(56)

$$U(\{p_N(z, v)\}, z) = \max_{C_T(s), C_N(s), p_T} \sum_v g(v) \left[\alpha \log C_T(s) + (1-\alpha) \log C_N(s) - b \left(C_T(s) + C_N(s)/A\right)\right]$$

subject to

$$C_T(s) = \frac{1}{p_T}$$

$$C_N(s) = \frac{1-\alpha}{b} \frac{p_T}{p_N(s)} \text{ for each } s = (z, v)$$

where we have assumed the cash-in-advance constraint is binding, which we have argued will always occur in a Markov equilibrium. Because policy in the monetary union is chosen to maximize a equally-weighted sum of utility of all countries, the weights $g(v)$ in the summation in (56) represent the fraction of all countries with idiosyncratic realization v. Since this fraction also represents the probability that an individual country will experience an idiosyncratic realization v, the utility function U is also the expected utility for any individual country.

Solving this problem gives the best response of the monetary authority to any given $\{p_N(s)\}$ and z which can be written as $p_T = p_T^{Union}(\{p_N(s)\}, z)$. It turns out that this best response only depends on a simple summary statistic of the distribution of nontraded goods prices namely $E(1/p_N(s)|z)$, the conditional mean of the inverse of these prices and we can write the best response as

$$(57)\quad p_T^{Union}(\{p_N(s)\}, z) = F\left(E\left(\frac{1}{p_N(s)}|z\right)\right)$$

for the same function F defined under flexible exchange rates in (46). Substituting that wages equal the price of the traded goods, the price chosen by each monopolist is

$$(58)\quad p_N(s) = \frac{1}{\theta(s)}\frac{p_T}{A}.$$

Hence, in equilibrium, the price of traded goods must satisfy the fixed point problem

$$(59)\quad p_T(z) = F\left(E\left(\frac{\theta(s)A}{p_T(z)}|z\right)\right)$$

Using this value it is easy to solve for the rest of the allocations from the constraints.

Lemma 4. Under (8) and (47), the allocations in the Markov equilibrium in a monetary union satisfy

$$(60)\quad C_T(s) = \frac{\alpha}{b} - \frac{1-\alpha}{b}\left(1 - \sum_s h(s)\theta(s)\right)$$

$$(61)\quad C_N(s) = \frac{1-\alpha}{b}\theta(s)A$$

and $L(s) = C_T(s) + C_N(s)/A$.

C. Comparing Welfare

We turn now to a comparison of welfare under flexible exchange rates with that under a monetary union. We begin with the case in which there are only markup shocks and then study the impact of productivity shocks. Briefly, as long as the idiosyncratic volatility of productivity shocks is sufficiently small relative to that of markup shocks then a monetary union is preferred to flexible exchange rates.

Only Markup Shocks

Consider first the case with only markup shocks. In Figure 2 we superimpose on our earlier Figure 1 the best response of the monetary authority in the union graph the best response of the monetary authority and the price chosen by each monopolist. Here for each aggregate state z, the monetary authority chooses a single price $p_T(z)$ that does not respond to any of the idiosyncratic shocks v. Thus, the equilibrium consumption of traded goods varies less than under flexible exchange rates.

More precisely, comparing (48)–(49) and (60)–(61), the outcome under flexible exchange rates differs from the outcome in a monetary union only in terms of the consumption of the traded good and the labor needed to produce it. In particular, from (48) and (60) it follows that the expected consumption of traded goods is constant in both regimes but the traded goods consumption is more volatile under flexible exchange rates. Hence, because of concavity of preferences over traded consumption goods, the ex-ante welfare associated with the Markov equilibrium in a monetary union is higher than under flexible exchange rates.

Proposition 3. Under (8) and (47, with only markup shocks the ex ante utility in the Markov equilibrium for a monetary union is strictly higher than the ex ante utility in the Markov equilibrium with flexible exchange rates.

Proof. Plugging the formulas for tradable and non-tradable consumption under the two regimes, (48)–(49) and (60)–(61), in the objective function and simplifying gives that the difference in value for a given initial aggregate state z between the welfare in a union and that under flexible exchange rates is given by

$$(62)\quad K\left(E\left[\theta|z\right]\right) - E\left[K(\theta)|z\right]$$

where the function $K(\theta) = \alpha \log\left((1-\alpha)(\theta-1)+\alpha\right)$. Since the function $K(\theta)$ is strictly concave in θ we know that the difference (62) is nonnegative and is strictly positive whenever there is variability in the idiosyncratic shock ν. $Q.E.D.$

Consider now how money growth and inflation compare in the two regimes. Under (**??**) the expression for the money growth rate reduces to $\mu^{flexible}(s) = G(\theta(s))$ under flexible exchange rates, and to $\mu^{union}(z) = G(E(\theta|z)$ in a monetary union where

$$G(\theta) = \frac{\beta\alpha}{[(1-\alpha)\theta - (1-2\alpha)]}.$$

Since G is a convex function of θ the expected value of money growth rate is higher under flexible exchange rates than in the union.

Consider next the inflation rates in the tradable and non-tradable sector from state s at one date to state s' at the next. Under flexible exchange rates these inflation rates are given by

$$\pi_T^{flexible}(s, s') = G(\theta(s)) \text{ and } \pi_N^{flexible}(s, s') = \frac{\theta(s)}{\theta(s')} G(\theta(s))$$

and in the union they are given by

$$\pi_T^{union}(s, s') = G(E(\theta|z)) \text{ and } \pi_N^{union}(s, s') = \frac{\theta(s)}{\theta(s')} G(E(\theta|z))$$

The convexity of G implies that in a monetary union inflation is not only less volatile than under flexible exchange rates but also is lower on average. This lower and less volatile inflation rate is beneficial because it results in distortions in the consumption in the tradable good that are on average lower and less volatile.

Only Productivity Shocks

Consider now the opposite extreme in which there are only productivity shocks. Here we show that the standard Mundellian forces are still operative so that a regime of flexible exchange rates are strictly preferred to one of a monetary union.

Proposition 4. Under (8) and (**??**) with only productivity shocks and no markup shocks welfare is higher under flexible exchange rates than in a monetary union.

There are two parts to the proof. The first part highlights the similarity in logic with the commitment case. We show that if the price of nontraded goods that confronts the monetary authority when it chooses its policy is the same under both regimes then welfare would be strictly higher under flexible exchange rates. The reason here is that the monetary authority is better able to adjust the price of traded goods to shocks. In particular, it is able to lower the relative price of nontraded to traded goods in the face of high productivity shocks and raise this relative price in the face of low productivity shocks. Doing so makes the marginal rate of substitution between nontraded and traded goods better track the marginal rate of transformation of these goods than is possible in the union.

The second part is more subtle. It shows that, in equilibrium, the price of nontraded goods that confronts the monetary authority under flexible exchange rates is actually lower than it is under a monetary union. A lower price of nontraded goods means that the economy is less distorted in terms of monopoly power and this feature tends to reinforce the benefits of flexible exchange rate. XXXX

Both Shocks

When we allow for both shocks we have two competing forces. The inability to react to the idiosyncratic markup shocks is a beneficial form of commitment and represents an advantage of the monetary union. In contrast, the inability to react to productivity shocks simply constrains the monetary authority from making marginal rates of substitution track corresponding marginal rates of tranformation and, hence, represents a cost. A union is preferable to flexible exchange rates when the commitment problem is relatively large, that is, the when variances of the idiosyncratic markup shocks are relatively large, compared to the standard Mundellian forces which are large when the variances of the idiosyncratic productivity shocks are large.

We illustrate this point first is a simple corollary to Proposition 3 and then with a simple numerical example.

Corollary. Consider an economy that satisfies (8) and (47) and the idiosyncratic volatility of the markup shocks is strictly positive. Then the ex ante utility in the Markov equilibrium with a monetary union is strictly higher than the ex ante utility in the Markov equilibrium with flexible exchange rates as long as the variability of idiosyncratic productivity shocks is sufficiently small.

This corollary immediately follows from Proposition 3 and continuity of the equilibrium values in the parameters of the model. Thus, when the monetary authority cannot commit to its policy, a group of ex-ante homogeneous countries can gain from joining a union when the variability of ex-ante idiosyncratic shock is large relative to the variability of ex-post idiosyncratic shock.

We illustrate this corollary in Figure 3. In this figure we plot the expected value of the Markov equilibrium under the two regimes as we vary the relative volatility of the idiosyncratic component of the productivity shock in the non-tradable sector. We parameterize the model by considering a simple case with no aggregate shocks: $\theta(\nu_1) \in \{1.1, 1.2\}$ and $A(v_2) \in \{1-\varepsilon, 1+\varepsilon\}$ where $g^1(\nu_1)$ and $g^2(\nu_2)$ are uniform and we vary $\varepsilon \geq 0$. As shown in Proposition 3, when $\varepsilon = 0$ the expected value of a Markov equilibrium for a country in a monetary union is higher than the what the same country can attain under flexible exchange rates. As ε increases and the variability of the idiosyncratic component of the productivity shocks increases the losses of monetary independence gets larger: the country cannot accommodate the idiosyncratic shocks in the tradable sector and cannot increase production of non-traded good when its productivity is high. Hence, there is a cutoff on the relative variabilities of the idiosyncratic shocks: the union if preferred if and only if the relative variability of the markup shocks are sufficiently high relative to that of productivity shocks.

4. The Optimal Configuration of Unions

So far we have assumed that all countries are ex-ante symmetric and studied their incentives to join a monetary union rather than to stay under a regime of flexible exchange rates. Here we allow countries to be ex-ante heterogeneous. In particular, we imagine there two groups of countries N and S, with a measure $\bar{m}^N$ of Northern countries and a measure of Southern countries $\bar{m}^S$. Here we focus on an economy with only markup shocks and we let the markup shocks in the North be $\theta^N(s_t)$ and the markup shocks in the South be $\theta^S(s_t)$. These shocks are realized at the beginning of the period (and, as before, we drop the subscript 1 denoting the beginning of the period for simplicity).

We consider a situation in which the Northern countries are already in a union of the type we discussed in the last section. These Northern countries are now considering whether it is in their best interest to admit countries from the South and if so how many such countries to admit. The Northern countries understand that if they let in a measure m^S of Southern countries then the policy in the union will be to maximize a weighted average of the utility of the Northern and the Southern countries where the weights are

$$(63)\quad \lambda^N = \frac{\bar{m}^N}{\bar{m}^N + m^S}, \lambda^S = \frac{m^S}{\bar{m}^N + m^S}.$$

and clearly the resulting vector λ satisfies $\lambda^i \in [0,1]$ and $\lambda^N + \lambda^S = 1$. For now, we assume that the Southern countries are originally under flexible exchange rates and will join the union only if they receive higher utility in the union than under flexible exchange rates. We show below that if the Southern countries are originally in a union of their own we get similar results.

To determine the size of the union we begin by solving for the Markov equilibrium and the welfare of individual countries given a particular composition of the union. We then ask what composition maximizes the welfare of the Northern countries given that the Southern countries that join the union must be made better off by doing so.

First, given a particular composition of the union (λ^N, λ^S) we solve for the allocations and welfare for Northern and Southern countries. Note that here the distribution of money holdings is degenerate since the analog of Lemma 2 applies and the cash in advance constraint binds in both the North and the South. The problem for the union in the aggregate state z given $\left\{p_N^N(s), p_N^S(s)\right\}$, is $U(\left\{p_N^N(s), p_N^S(s)\right\}, z) =$

$$\max_{C_T^i(s), C_N^i(s), p_T} \sum_{i=N,S} \lambda^i \sum_v g(v) \left[\alpha \log C_T^i(s) + (1-\alpha) \log C_N^i(s) - b\left(C_T^i(s) + \frac{C_N^i(s)}{A^i}\right)\right]$$

subject to for all v

$$C_T^i(s) = \frac{1}{p_T}$$

$$C_N^i(s) = \frac{1-\alpha}{b}\frac{p_T}{p_N^i(s)}$$

where $s = (z, v)$. Notice that, as remarked earlier, the summation over v in the objective function is not because there is any uncertainty to be realized but rather the simply represents the utilitarian welfare weights. The resulting allocations are summarized in the next lemma.

Lemma 4. Under (8) and constant productivity A^N, A^S, the allocations in the Markov equilibrium in a monetary union satisfy for $i = N, S$

$$(64)\quad C_T^i(s,\lambda) = c_0 + \frac{1-\alpha}{b}\left(\sum_{i=N,S}\lambda^i\sum_{\tilde{v}} g(v)\theta^i(\tilde{s})\right)$$

$$(65)\quad C_N^i(s,\lambda) = \frac{1-\alpha}{b}\theta^i(s)A^i$$

and $L^i(s,\lambda) = C_T^i(s,\lambda) + C_N^i(s,\lambda)/A^i$ where $c_0 = (2\alpha-1)/b$, $s = (z,v)$ and $\tilde{s} = (z,\tilde{v})$

Note that the allocation of traded goods $C_T^i(s,\lambda)$ does not depend on the particular realization of the idiosyncratic shock v in $s = (z,v)$ but only only the distribution. Also note that the allocation of nontraded goods does not depend on the welfare weights λ.

This lemma immediately gives that the expected welfare of the both Southern and Northern countries for a vector of welfare weights λ.

$$(66)\quad W^i(\lambda) = \alpha E\log C_T^i(s,\lambda) + (1-\alpha)E\log C_N^i(s,\lambda) - bEL^i(s,\lambda)$$

We then turn to asking what is the optimal measure of Southern countries to admit to the union. Formally, this problem is to choose $\lambda = (\lambda^N, \lambda^S)$

$$(67)\quad \max_{\lambda} W^N(\lambda)$$

subject to a feasibility constraint

$$(68)\quad \lambda^N \geq \frac{\bar{m}^N}{\bar{m}^N + \bar{m}^S}$$

and the participation constraint of Southern countries $W^S(\lambda) \geq W_{flex}^S$ where W_{flex}^S is defined from the allocations under flexible exchange rates given in Lemma 2. We will assume that $\bar{m}^S$ is sufficiently large compared to $\bar{m}^N$ so that the feasiblity constraint does not bind.

Using the logic in Proposition 3 it is straightforward to prove that if the Southern countries are on average more distorted and have more variable distortions then they always prefer joining the union with the North than a staying on their own. Formally, if

$$(69) \quad E\theta^S \leq E\theta^N \text{ and } var(\theta^S) \geq var(\theta^N)$$

the participation constraint above is slack. We will assume that (69) holds in all of our comparisons below.

Consider now how the optimal composition of the union from the North's perspective changes as the average distortions in the South vary. For intuition's sake suppose first there is no uncertainty and the South is strictly more distorted than the North in that $\theta^S < \theta^N$. Clearly, allowing any Southern countries to join the union only exacerbates the time inconsistency problem so it is not optimal to allow any Southern countries to join. Specifically, since the consumption of nontraded goods is independent of the composition of the union and since the consumption of traded goods is distorted downward relative to the efficient allocation, the optimal composition maximizes the consumption of traded goods, which here becomes

$$(70) \quad C_T^i(\lambda) = c_0 + \frac{1-\alpha}{b}\left(\lambda^N\theta^N + \lambda^S\theta^S\right)$$

Since $\theta^S < \theta^N$ this consumption is decreasing in the fraction of Southern countries in the union so that the optimal composition has $\lambda^S = 0$.

Next, suppose that markups are stochastic. As the South get more distorted we show that the optimally admits fewer Southern countries. Specifically, consider two specifications for the distortions in the South: the original process $\theta^S(s)$ and another more distorted one, say $\hat{\theta}^S(s) = \theta^S(s) - \Delta(s)$ for some vector with $\Delta(s) \geq 0$ for all s. Let λ^S and $\hat{\lambda}^S$ denote the optimal composition in the two specifications.

Proposition 5. Under (69), $\hat{\lambda}^S \leq \lambda^S$ if $\Delta(s) \geq 0$ for all s. That is, the greater the average distortion in the Southern countries, the fewer the North admits into the union.

The idea here is essentially the same as in the deterministic case.

Consider now how the optimal composition of the union from the North's perspective changes as the variability of the distortions in the South changes as the correlation of the Southern and Northern distortions changes. For simplicity we now suppose that the mean distortion in the North and the South are the same, in that $E(\theta_N) = E(\theta_S)$. With this assumption it follows that the expected value of labor in the welfare function (66) is independent of λ. Since the consumption of nontraded goods is also independent of λ, optimal composition of the union maximizes the expected utility from the consumption of traded goods, namely $E \log C_T^i(s, \lambda)$. If we define the random variable $x(s, \lambda) = \left(\sum_{i=N,S} \lambda^i \sum_{\tilde{v}} g(v)\theta^i(\tilde{s})\right)$, the problem reduces to one of maximizing the expected value of $\log(c_0 + \frac{1-\alpha}{b}x)$. Since this expected value is a concave value of x the optimal composition of the union minimizes the variance of x that is λ minimizes the variance of $x(s, \lambda)$, that is,

$$\min_{\lambda^N, \lambda^S} \left(\lambda^N\right)^2 \sigma_N^2 + \left(\lambda^S\right)^2 \sigma_S^2 + 2\lambda^N \lambda^S \rho \sigma_N \sigma_S$$

where $\lambda^N + \lambda^S = 1$, σ_i^2 is the variance of $\theta^i(s)$ and ρ is the correlation of $\theta^N(s)$ and $\theta^S(s)$. The optimal composition of the union is then given in the next proposition.

Proposition 6. Assume $E(\theta_N) = E(\theta_S)$ and (69). If $\rho\sigma_S < \sigma_N$ then

$$(71) \quad \lambda^S = \frac{\sigma_N^2 - \rho\sigma_N\sigma_S}{\sigma_S^2 + \sigma_N^2 - 2\rho\sigma_N\sigma_S}$$

if not $\lambda^S = 0$. Hence, the measure of Southerners admitted to the union decreases with their volatility ($\partial\lambda^S/\partial\sigma_S \leq 0$). The measure of Southerners admitted to the union decreases with their correlation with the Northern shocks ($\partial\lambda^S/\partial\rho \leq 0$).

To get a feel for what this result implies, assume first that the Southern and Northern shocks are uncorrelated so that $\rho = 0$. Then if the variances of these shocks are equal, it is optimal to have a union in which half of its members are Southern. As the variance of the Southern shocks is increased to, say, double that of the Northern shocks then it is optimal to have only one-third of the union be from the South.

Note that here if the South both more distorted and more volatile than the North in that it satisfies (69), then the North will never allow the South to be more than one half of the union. (tighten Proof)

MORE?

Now suppose we ask what configuration of unions does the model predict. We will focus on configurations that are *stable* in the sense that there is no deviation by a group of countries to form their own union that make all of the members of the deviating group weakly better off and at least one type of them strictly better off. Clearly, if two unions have the same welfare weights then they will pursue identical policies and hence their exchange rates will be fixed. Thus, we can consider such a pair of unions as simply being one union. [3]

We claim that the there is a unique stable configuration of unions: a mixed North-South union with the mixture chosen as above, say at $\hat{\lambda}$, and a pure Southern union. Clearly, no measure of Northern and Southern countries can defect from this equilibrium and form a union which both defecting groups strictly prefer to the one proposed. Notice that for any $\lambda \neq \hat{\lambda}$ a measure of Northern and Southern countries can defect to one with a mixture $\hat{\lambda}$ and make the defectors strictly better off. As we did above, here we are assuming that there are sufficiently many more Southern countries than Northern countries so that this allocation is feasible. We summarize this discussion as follows.

Proposition 7. A mixed North South union with λ chosen to solve (**??**) and a pure Southern union consisting of the remainder Southern countries is the unique stable configuration of unions.

We briefly consider a more general case with three groups of countries: North, Middle, and South. Let these groups be ranked in a pecking order in that mean distortions and volatilities be increasing from North to South while the volatilities are increasing from North to South.

Consider a configuration with three unions. In the first union the weights $\lambda_1 = (\lambda_1^N, \lambda_1^M, \lambda_1^S)$ maximize $W^N(\lambda)$ while in the second union the weights $\lambda_2 = (0, \lambda_2^M, \lambda_2^S)$ maximize $W^M(\lambda)$ subject to the restriction that $\lambda^N = 0$. In the third union the weights are $\lambda_3 = (0, 0, 1)$. It is straightforward to construct the masses of countries (m_i^k) in each of these groups. In the construction we assume that the measure of countries is such that $\bar{m}^M/\bar{m}^N$ and $\bar{m}^S/\bar{m}^M$ are sufficiently large so that the configuration we construct is feasible.

[3] Let $\{m_i\}_{i=1}^I$ with $m_i = (m_i^N, m_i^S)$ with $\sum_i m_i^N = \bar{m}^N$ and $\sum_i m_i^S = \bar{m}^S$ and $m_i^N + m_i^S > 0$ for each i be a *partition* of the union and let $V_i = (V^N(m_i), V^S(m_i))$ be the associated welfares. A configuration $\{m_i\}_{i=1}^I$ is stable if there does exist a deviating group of countries $\{\hat{m}_i\}_{i=1}^I$ with $\hat{m}_i \leq m_i$ such that $V^N(\sum \hat{m}_i) \geq V_i^N$ for all i such that $\hat{m}_i^N > 0$ and $V^S(\sum \hat{m}_i) \geq V_i^S$ for all i such that $\hat{m}_i^S > 0$ where at least one of these previous inequalities is strict. Note: here we are making all the deviators go to one new island, but that is not restrictive.

Proposition 8. The configuration $\lambda_1, \lambda_2, \lambda_3$ given above is the unique stable configuration of unions.

5. Anchor-Client Peg

So far we have considered the formation of unions in economies in which either all the monetary authorities can commit or all the monetary authorities cannot commit. Alesina and Barro (2002) and Alesina, Barro, Tenreyro (2003) consider a case in which some countries can commit and some cannot. They think of the countries with commitment as potential *anchor* countries and the countries without commitment as the potential *client* countries. The key assumption they make is that the policies of the anchor countries do not change as various clients joint a monetary union with them.

Consider a version of our model in which there is a given client country with some given set of stochastic processes for its shocks and a larger number of potential anchors each with its own stochastic processes for shocks. Now, let us start by considering what is the ideal anchor for a given client. The answer is immediate: the ideal anchor is the country that follows the policies that the client country would follow if it had commitment. Such an ideal anchor would be one in which the realization of all shocks (including "idiosyncratic" shocks) are identical. Obviously, when the client adopts the policy of such an ideal anchor it achieves its Ramsey welfare level and it cannot do better.

We turn now to the optimal choice of an anchor when the set of potential anchors may not include such an ideal anchor. Let $\{\theta(s_{1t}), A(s_{2t})\}$ denote the stochastic process of the client and let $\{\theta_i(s_{1t}), A_i(s_{2t})\}$ for $i = 1, \ldots, I$ denote the stochastic processes of the potential anchor. Then the following proposition holds.

Proposition 8. The optimal anchor country i^* for the given client country with stochastic process $\{\theta(s_{1t}), A(s_{2t})\}$ satisfies

$$\min_i \log\left(E\left[\frac{A_i\left(s_{2t}\right)}{A\left(s_{2t}\right)}\right]\right) - E\left[\log\frac{A_i(s_{2t})}{A\left(s_{2t}\right)}\right].$$

Notice that this proposition holds for general specifications of the stochastic processes for the client and the anchor. If we assume the processes for productivity shocks have the form $A_i(s_{2t}) = A_z(z_{2t})A_{vi}(v_{2t})$ and $A(s_{2t}) = A_z(z_{2t})A_v(v_{2t})$ so that the anchors and the client have a common aggregate component to productivity shocks then the optimal anchor i^* solves

$$\min_i \log \left(E \left[A_{vi} \left(v_{2t} \right) \right] \right) - E \left[\log A_{vi}(s_{2t}) \right].$$

Hence, when the idiosyncratic components of the productivity shocks A_{vi} are log normal variances σ_{vi}^2 it is optimal to pick the anchor with the lowest variance of idiosyncratic shocks σ_{vi}^2.

6. Criteria in Terms of Endogenous Variables

[need to talk about micro data; general message: need to take a stand on which policies are followed]

So far we have stated our criterion in terms of properties of the stochastic processes for productivity and markup. A large empirical literature tried to assess whether certain countries are good candidates to form a union by looking at the behavior of observables such as the idiosyncratic components output and real exchange rates. The standard view in the literature is that countries are poor candidates to form a monetary union if the variances of the idiosyncratic components of output and real exchange rates are large.

In this section we argue that viewed through the lens of our model, this standard view can be misleading: both with and without commitment, even when the variances of the idiosyncratic components of output and real exchange rates are both high, it may be desirable to form a union. It turns out that under commitment the most informative statistic is the variance of the rate of growth of nominal exchange rates: when this statistic is low, forming a union is desirable, when it is not, forming a union is not. Without commitment the most informative statistic is the variance of the idiosyncratic component of output relative that of real exchange rates: when this statistic is high forming a union is desirable, when it is low it is not.

Our finding is also related to Dornbusch's

The first step in expressing our criteria in terms of endogenous variables is to obtain expressions for them in terms of the shocks. To do so, note that the real exchange rates, the log of real output and the change in nominal exchange rates all relative to the world average can be expressed approximately in log-deviation form

$$\begin{aligned}
(72) \qquad \log q(s) &= (1-\alpha)\log p_N(s)/p_T(s) - (1-\alpha)E_v\left[\log p_N(z,v)/p_T(z,v)\right] \\
\log y\,(s) &= \alpha \log C_T\,(s) + (1-\alpha)\log C_N\,(s) - E_v\left[\alpha \log C_T\,(z,v) + (1-\alpha)C_N\,(z,v)\right] \\
\log E(s')/E(s) &= \left[\log P_T\,(s') - E_v \log P_T(z',v')\right] - \left[\log P_T\,(s) - E_v \log P_T(z,v)\right].
\end{aligned}$$

In order to make clear the role of the idiosyncratic components of the shocks, we assume in what follows that the productivity shocks and the markup shocks can be expressed as multiplicative functions of an aggregate component and an idiosyncratic component and the two components are independent of each others. Specifically, we assume that $A\,(z,v) = A_z\,(z)\,A_v\,(v)$ and $\theta\,(z,v) = \theta_z\,(z)\,\theta_v\,(v)$.

Consider first the commitment case in a symmetric environment. Since $p_N/p_T = 1/\left(A(s)\theta(s)\right)$, $C_T = \alpha/b$ and (26), using our formulation of the shocks we have that

$$\log q(s) = -(1-\alpha)\left[\log \theta_v(v) + \log A_v(v)\right] \tag{73}$$

$$\log y\,(s) = (1-\alpha)\left[\log \theta_v(v) + \log A_v(v)\right] \tag{74}$$

$$\log E(s')/E(s) = \log A_v(v'). \tag{75}$$

Consider now using observations from a group of countries observed when following the commitment policies under flexible exchange rates to determine whether they should form a union. From Proposition 2, we know that the welfare losses of forming a union depend only on the idiosyncratic variance of $1/A$. We now show how we can use observables to obtain estimates of the idiosyncratic variance of $1/A$. From (73) and (74) we see that we cannot disentangle the role of productivity and markup shocks from information on the real exchange rate and output so that these variables cannot by themselves be used to pin down the idiosyncratic variance of $1/A$, rather they can only be use to form an upper bound on this variance.

The standard view can be misleading when the variances of output and real exchange rates are both high. To see how, suppose, for example, that a group of countries has high variance of real exchange rates and output because they have high variability of markup shocks but no variability of idiosyncratic productivity shocks. It is very desirable for this group of countries to form a union but, under the standard view, they would be regarded as poor candidates for a union.

Our model implies that the key statistic to use to determine the desirability of a union is the variability of the growth rate of its nominal exchange rate. To see why, note from (75) that the variance of the nominal exchange rate can be used directly to obtain the idiosyncratic variance of $1/A$.

Consider now the case without commitment. Here again we begin by obtaining expressions for the endogenous variables in terms of the shocks. Taking a log-linear approximation to the Markov equilibrium outcomes under flexible exchange rates we obtain

$$(76) \quad \log q(v) = (1-\alpha)\log\theta_v(v) - (1-\alpha)\phi\log A_v(v)$$

$$(77) \quad \log y(v) = -\left[\frac{\alpha}{1-\phi}+1\right]\log\theta_v(v) + (1-2\alpha)\phi\log A_v(v)$$

where ϕ is given by $\phi \equiv -F'/\left(Fp_N A\right) > 0$ where F is defined in (46) and ϕ is evaluated at the deterministic steady state [4]. Clearly, (76) and (77) can be solved to express the variances of the idiosyncratic shocks in terms of the variances of the endogenous variables. Comparing (73) and (74 with (76) and (77) we see that the map between the endogenous variables and the shocks differs depending upon the policies being followed by the monetary authority. These policies, of course, are different between a regime of commitment and one of no commitment. Thus, the question of the desirability of

and (To evaluate the desirability of joining a union we then substitute these expressions back into the welfare function. Figure xx we graph the iso-welfare lines

Here we cannot obtain a closed form expression solution for the welfare gains of joining union whenever productivity shocks are not deterministic. We then obtain an approximation

[4] The deterministic steady state is given by $p_N = p_T/\theta A$ and

$$p_T = \frac{b}{\alpha - (1-\alpha)(1-\theta)}$$

of the welfare criterion by considering a second-order log-linear approximation of the objective function and a first order log-linear approximation of the decision rules in both regimes adjusting the mean as in Kim and Kim. In particular, we make sure that under our approximation is exact for the case in which we have a closed form solution for prices, allocation and welfare (see Lemma 4 and Proposition 3). In particular, we have that the welfare gains of forming a monetary union are given by

$$\frac{\sigma_\theta^2}{2}\left[\alpha\frac{\phi}{1-\phi}\right]-\frac{\sigma_A^2}{2}\left[(1-\alpha)-(1-\alpha)(\phi^2-2\phi+1)-\alpha\phi\right]$$

σ_θ^2 and σ_A^2 are the idiosyncratic variance of $\log\theta$ and $\log A$. Check that $\left[(1-\alpha)-(1-\alpha)(\phi^2-2\phi+1)-\alpha\phi\right.$
0. The first term represents the commitment gains of a monetary union, the second term stands in for the losses associated with the inability to repsond to productivity shocks. Consistently with Figure 3, the welfare gains of forming a monetary union without commitment are increasing in the idiosyncratic variability of markup shocks and decreasing in the idiosyncratic variability of productivity shocks.

Using a first order log-linear approximation, we obtain the following expressions for σ_θ^2 and σ_A^2 in terms of observables:[5]

$$\begin{bmatrix}\sigma_A^2\\ \sigma_\theta^2\end{bmatrix}=\begin{bmatrix}var\left(\log A(v_2)\right)\\ var\left(\log\theta(v_1)\right)\end{bmatrix}=\begin{bmatrix}\left(\frac{\alpha}{1-\phi}+1\right)^2 & (1-2\alpha)^2\phi^2\\ \left(\frac{\alpha\phi+(1-\alpha)}{1-\phi}\right)^2 & \alpha^2\phi^2\end{bmatrix}^{-1}\begin{bmatrix}var\left(\ln Y(s)-\ln\bar{Y}(z)\right)\\ var\left(\ln p(s)-\ln\bar{p}(z)\right)\end{bmatrix}$$

So???

Two points to make: when going from commitment to Markov:

Criterion without commitment differs for two reasons:

i) welfare criterion in terms of shocks is different

ii) mapping from observables to shocks changes

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[5]Need to make sure that

$$\Xi=\begin{bmatrix}\left(\frac{\alpha}{1-\phi}+1\right)^2 & (1-2\alpha)^2\phi^2\\ \left(\frac{\alpha\phi+(1-\alpha)}{1-\phi}\right)^2 & \alpha^2\phi^2\end{bmatrix}$$

is not singular.

7. Conclusion

We have presented a new argument for why forming a monetary union among symmetric countries may be desirable.

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8. Appendix

Here we provide some omitted proofs.

A. Derivation of the Ramsey Outcome and Proof of Lemma 3

Here we derive the Ramsey outcome under (8). Consider an even more relaxed version of the relaxed Ramsey problem (**??**) by dropping (**??**). Letting $\xi(s^{t-1}, s_{1t})$ be the multiplier associated with (**??**), dividing the first order condition for $C_N(s^t)$ by that for $L(s^t)$ gives us

$$(78) \quad \frac{1-\alpha}{bC_N(s^t)} = \frac{1}{A(s_{2t})}\left[1 + \xi(s^{t-1}, s_{1t})\frac{1}{\theta(s_{1t})}\right]$$

which can be solved for $C_N(s^t)$ to yield:

$$(79) \quad C_N(s^t) = \frac{A(s_{2t})(1-\alpha)}{b}\frac{1}{1+\xi(s^{t-1}, s_{1t})/\theta(s_{1t})}$$

Clearly the consumption of nontraded goods is given by

$$(80) \quad C_T(s^t) = \frac{\alpha}{b}$$

Then, substituting (79) into (**??**) for all s^{t-1}, s_1 and solving for $\xi(s^{t-1}, s_1)$ we get

$$(81) \quad \xi(s^{t-1}, s_{1t}) = \theta(s_{1t})\left[\theta(s_{1t}) - 1\right]$$

Thus consumption of nontraded is given by

$$(82) \quad C_N(s^t) = \frac{A(s_{2t})(1-\alpha)}{b}\frac{1}{1+\theta(s_{1t})-1} = \frac{A(s_{2t})}{\theta(s_{1t})}\frac{(1-\alpha)}{b}$$

and obviously

$$(83) \quad L(s^t) = C_T(s^t) + \frac{C_N(s^t)}{A(s_{2t})}$$

We next show that this allocation can be implemented as a competitive equilibrium, proving Lemma 3 in the text.

In normalized prices and money the cash in advance constraint is

$$p_T(s^t)C_T(s^t) \leq 1.$$

$$\frac{p_T(s^t)}{p_N(s^t)} = \frac{A(s_{2t})}{\theta(s_{1t})}$$

$$p_N(s^t) = \theta(s_{1t})$$

$$(84) \quad P_N(j, s^{t-1}, s_{1t}) = \theta(s_{1t}) \frac{\sum_{s_{2t}} Q_t(s^t) C_{Nt}(s^t) \frac{W_t(s^t)}{A(s_t)}}{\sum_{s_{2t}} Q_t(s^t) C_{Nt}(s^t)}.$$

with log becomes

$$p_N(s^t) = \theta(s_{1t}) \frac{\sum_{s_{2t}} Q_t(s^t) C_{Nt}(s^t) \frac{p_T(s^t)}{A(s_t)}}{\sum_{s_{2t}} Q_t(s^t) C_{Nt}(s^t)}$$

$$= \theta(s_{1t}) \frac{\sum_{s_{2t}} \pi(s_{2t}|s^{t-1}, s_{1t}) U_N(s^t) C_{Nt}(s^t) \frac{p_T(s^t)}{A(s_t)}}{\sum_{s_{2t}} \sum_{s_{2t}} \pi(s_{2t}|s^{t-1}, s_{1t}) U_N(s^t) C_{Nt}(s^t)}$$

since $U_N = (1-\alpha)/C_N$ so becomes

$$= \theta(s_{1t}) \frac{\sum_{s_{2t}} \pi(s_{2t}|s^{t-1}, s_{1t}) U_N(s^t) C_{Nt}(s^t) \frac{p_T(s^t)}{A(s_t)}}{\sum_{s_{2t}} \sum_{s_{2t}} \pi(s_{2t}|s^{t-1}, s_{1t}) U_N(s^t) C_{Nt}(s^t)}$$

$$p_N(s^t) = \theta(s_{1t}) \sum_{s_{2t}} \pi(s_{2t}|s^{t-1}, s_{1t}) \frac{p_T(s^t)}{A(s_t)}$$

$$p_T(s^t)C_T(s^t) \leq 1.$$

$$(85) \quad p_T(s^t) = \frac{A(s_{2t})}{\theta(s_{1t})} p_N(s^t)$$

$$(86) \quad C_T(s^t) = \frac{\alpha}{b} \text{ and } C_N(s^t) = \frac{1-\alpha}{b} \frac{A(s_{2t})}{\theta(s_{1t})}$$

so must have

$$\frac{A(s_{2t})}{\theta(s_{1t})} p_N(s_{1t}) \frac{\alpha}{b} \leq 1 \text{ for all } s_{2t}$$

$$(87) \quad p_N(s_{1t}) \leq \frac{b}{\alpha} \frac{\theta(s_{1t})}{A(s_{2t})}$$

so can have CIA slack everywhere or CIA binds at some $A(s_{2t})$. So pick that (87) holds with equality at the highest $A(s_{2t})$. For all $A' < \max A(s_{2t})$ the right side is bigger so the

$$p_N(s_{1t}) = \frac{b}{\alpha}\frac{\theta(s_{1t})}{\max A(s_{2t})} < \frac{b}{\alpha}\frac{\theta(s_{1t})}{A'} \text{ for all } A' < \max A(s_{2t})$$

Then the following uniquely pins down $p_T(s^t)$.

$$(88)\quad p_T(s^t) = \frac{A(s_{2t})}{\theta(s_{1t})} p_N(s^t)$$

then we form

Proof of Lemma 3. Consider implementing $\{C_T(s^t), C_N(s^t), L(s^t)\}$ given by (80)–(83) as a competitive equilibrium. We construct the prices so that the cash-in-advance constraint holds with equality holds at the highest level of productivity of the nontraded goods and is slack at all other shocks. (Of course, one could have the cash in advance slack at all shocks and this would shift the prices down for the same money supplies). For all t, s^t, recursively construct prices normalized by the beginning of the period money holdings, $p_T(s^t) = P_T(s^t)/M(s^{t-1})$ and $p_N(s^{t-1}, s_{1t}) = P_N(s^{t-1}, s_{1t})/M(s^{t-1})$ and money growth rate as:

$$(89)\quad p_N(s^{t-1}, s_{1t}) = \min_{s_2}\left\{\frac{b}{\alpha}\frac{\theta(s_{1t})}{A(s_{2t})}\right\} = \frac{b}{\alpha}\frac{\theta(s_{1t})}{\max A(s_{2t})}$$

$$(90)\quad p_T(s^t) = \frac{A(s_{2t})}{\theta(s_{1t})} p_N(s^{t-1}, s_{1t})$$

$$(91)\quad \frac{M(s^t)}{M(s^{t-1})} = \beta \sum_{s^{t+1}} h(s^{t+1}|s^t) \frac{\alpha}{p_T(s')C_T(s')} \Big/ \frac{(1-\alpha)}{C_N(s^t)p_N(s^{t-1}, s_{1t})} = \beta$$

The allocations $\{C_T(s^t), C_N(s^t), L(s^t)\}$ and the process $\{P_T(s^t), P_N(s^{t-1}, s_{1t}), M(s^t), W(s^t)\}$ obtained from (89)–(91) where we let $W(s^t) = P_T(s^t)$ is a competitive equilibrium outcome. First notice that the sufficient conditions for households optimality are satisfied. $W(s^t) = P_T(s^t)$ and (80)gives (13); combining (90), (89), (82) and using $W(s^t) = P_T(s^t)$ gives (12); (91), (80), (82), (90), and (89) imply (14); finally notice that (9) is satisfied by substituting (90) and (80) in the cash-in-advance constraint. Nominal interest rates $\{r_t(s^t)\}$ and state-prices $\{Q_t(s^t)\}$ are given by (15) and (16). The constructed prices satisfy (**??**) because the allocations satisfy (**??**). Finally, market clearing follows from the feasibility of the allocations. *Q.E.D.*

We now turn to the Ramsey problem for a monetary union under (8). Consider the following relaxed problem:

$$(92)\quad \max \sum_t \sum_{s^t} \beta^t h(s^t) \left[\alpha \log\left(C_T(s^t)\right) + (1-\alpha)\log\left(C_N\left(s^t\right)\right) - b\left(C_T(s^t) + \frac{C_N(s^t)}{A(s_{2t})}\right)\right]$$

subject to (**??**) and

$$(93)\quad C_N(s^t) = C_N(s^{t-1}, s_{1t}, z_{2t}) \text{ for all } \nu_{2t}$$

where the last constraint imposes that $C_N(s^t)$ cannot vary with v_{2t}.and follows from (**??**).

After substituting the last constraint in the objective function, the first order condition for $C_N(s^{t-1}, s_{1t}, z_{2t})$ can be written as

$$(94)\quad \frac{1-\alpha}{C_N(s^{t-1}, s_{1t}, z_{2t})} = \left(1 + \xi(s^{t-1}, s_{1t})\right) \theta(s_{1t}) \sum_{\nu_2} g^2(\nu_{2t}) \frac{b}{A(s_{2t})}$$

where $\xi(s^{t-1}, s_1)$ is the multiplier on (**??**). The first order condition for $C_T(s^t)$ simply gives

$$(95)\quad C_T(s^t) = \frac{\alpha}{b}$$

Defining $\bar{X}(z_2) = \sum_{\nu_2} g^2(\nu_2)/A(s_2)$, we can solve (94) for C_N obtaining

$$(96)\quad C_N(s^{t-1}, s_{1t}, z_{2t}) = \frac{1-\alpha}{(1 + \xi(s^{t-1}, s_{1t}))\, \theta(s_{1t}) b \bar{X}(z_{2t})}$$

and substituting back into the labor market distortion constraint, (**??**), we can solve for the multiplier, obtaining:

$$(97)\quad \left(1 + \xi(s^{t-1}, s_{1t})\right) = \sum_{s_2} h^2(s_{2t}) \frac{1/A(s_{2t})}{\bar{X}(z_{2t})}$$

Plugging back the expression for $\xi(s^{t-1}, s_{1t})$ into (96) gives:

$$(98)\quad C_N(s^{t-1}, s_{1t}, z_{2t}) = \frac{1}{\theta(s_{1t})} \frac{1-\alpha}{b} \frac{1}{\bar{X}(z_{2t}) \sum_{\tilde{s}_2} h^2(\tilde{s}_2) \frac{1/A(\tilde{s}_2)}{\bar{X}(\tilde{z}_2)}}$$

and obviously

$$(99)\quad L(s^t) = C_T(s^t) + \frac{C_N(s^t)}{A(s_{2t})}$$

We now show that the allocations in (95), (98)–(99) can be implemented as a competitive equilibrium under a monetary union. In particular, we construct prices such that the cash-in-advance constraint holds with equality in all states. For all t, s^t, construct prices normalized by the beginning of the period money holdings, $p_T(s^t) = P_T(s^t)/M(s^{t-1})$ and $p_N(s^{t-1}, s_{1t}) = P_N(s^{t-1}, s_{1t})/M(s^{t-1})$ and money growth rate as follows:

$$(100)\quad p_N(s^{t-1}, s_{1t}) = \frac{b}{\alpha} \theta(s_1) \min_{z_2} \left\{\bar{X}(z_2)\right\} \sum_{s_2} h^2(s_2) \frac{1/A(s_2)}{\bar{X}(z_2)}$$

$$(101)\quad p_T(s^t) = \frac{A(s_{2t})}{\theta(s_{1t})} p_N(s^{t-1}, s_{1t}) = \frac{b}{\alpha} \frac{\min_{z_2} \left\{\bar{X}(z_2)\right\}}{\bar{X}(z_2)}$$

$$(102)\quad \frac{M(s^t)}{M(s^{t-1})} = \beta \frac{\bar{X}}{\bar{X}(z_2)}$$

The allocations $\{C_T(s^t), C_N(s^t), L(s^t)\}$ and the process $\{P_T(s^t), P_N(s^{t-1}, s_{1t}), M(s^t), W(s^t)\}$ obtained from (100)–(102) where we let $W(s^t) = P_T(s^t)$ is a competitive equilibrium outcome in a monetary union. First notice that the sufficient conditions for households optimality are satisfied. $W(s^t) = P_T(s^t)$ and (95)gives (13); combining (101), (100), (98) and using $W(s^t) = P_T(s^t)$ gives (12); (102), (95), (98), (101), and (100) imply (14); finally notice that (9) is satisfied by substituting (101) and (95) in the cash-in-advance constraint. Nominal interest rates $\{r_t(s^t)\}$ and state-prices $\{Q_t(s^t)\}$ are given by (15) and (16). The constructed prices satisfy (**??**) because the allocations satisfy (**??**). Finally market clearing follows from the feasibility of the allocations.

B. Definition Markov Union

The consumer's problem is the same as under flexible exchange rates except that the state is augmented to include S_H. Here the union-wide aggregate money growth rate γ defined as $\bar{M}/\bar{M}_{-1}$ is given by

$$\gamma = \int [\mu(x_G, S_G)m] \, d\lambda_G.$$

The problem for the monetary authority is

$$(103) \quad W(S_G) = \max_{\mu(x_G)} \int V(m, x_H, S_H) \, d\lambda_G$$

A *Markov equilibrium in a monetary union* consists of sticky price decision rules $\bar{p}_N(x_F, S_F)$, households decision rules $C_N(m_H, x_H, S_H)$, $C_T(m_H, x_H, S_H)$, $L(m_H, x_H, S_H)$, $m'_H(m_H, x_H, S_H)$, and value function $V(m_H, x_H, S_H)$, price rules $\bar{w}(x_H, S_H)$ and $\bar{p}_T(x_H, S_H)$, profit rules $\pi(x_H, S_H)$, such that i) in the current period and all future periods the flexible price firm and the household decision rules are optimal in that the flexible price firms' price rule satisfies

$$(104) \quad \bar{p}_T(x_H, S_H) = \bar{w}(x_H, S_H)$$

and the household decision rules are optimal for problem (31) and the value function V and the profit rule satisfies

$$(105) \quad \pi(x_H, S_H) = \left(p_N - \frac{\bar{w}(x_H, S_H)}{A(s_2)} \right) C_N(m, x_H, S_H)$$

ii) the sticky price firms' price rule satisfies

$$(106) \quad \bar{p}_N(x_F, S_F) = \theta(s_1) \frac{\sum_{s_2} h^2(s_2) U_N(m, x_H, S_H)) C_N(m, x_H, S_H) \bar{w}(x_H, S_H)/A(s_2)}{\sum_{s_2} h^2(s_2) U_N(m, x_H, S_H)) C_N(m, x_H, S_H)}$$

where (x_H, S_H) are induced from (x_F, S_F) from $\bar{\mu}$, iii) the market clearing conditions hold, $C_N(m, x_H, S_H) = A(s_2) L_N(x_H, S_H)$, $C_T(m, x_H, S_H) = L_T(x_H, S_H)$, $L(m, x_H, S_H) = L_N(x_H, S_H) +$

$L_T(x_H, S_H)$, as well as money market clearing in the current period

(107) $m'_H(m, x_H, S_H) = \dfrac{\mu(x_G, S_G)m}{\int \mu(x_G, S_G)m d\lambda_G}$

where x_H is induced from x_G by $\mu(x_G, S_G)$ and money market clearing in all future periods

(108) $m'_H(m, x_H, S_H) = \dfrac{\bar{\mu}(x_G, S_G)m}{\int \bar{\mu}(x_G, S_G)m d\lambda_G}$

where x_H is induced from x_G by $\bar{\mu}(x_G, S_G)$, iv) government decision rule and value function solve (103) and v) the fixed exchange rate constraint

(109) $\bar{p}_T(x_H, S_H) = \bar{p}_T(S_H)$ for all x_H, S_H

holds.

C. Proof of Lemma 2

Before we prove this lemma we show that under our utility function (8) the Markov problem can be greatly simplified. First we note that under this utility function that problem in a union can be written.

(110)

$$\bar{W}(S_G) = \max_{p_T, C_T(x_G), C_N(x_G), \mu(x_G)} \int \left[U\left(C_T(x_G), C_N(x_G), C_T(x_G) + \frac{C_N(x_G)}{A(x_G)} \right) \right] d\lambda_G + \beta \sum_s h(s') \bar{W}(S'_G)$$

subject to

(111) $C_N(x_G) = \dfrac{1-\alpha}{b} \dfrac{p_T}{p_N(x_G)}$

(112) $C_T(x_G) = \min \left\{ \dfrac{m(x_G)}{p_T}, \dfrac{\alpha}{b} \right\}$

(113) $\gamma \dfrac{b}{p_T} = \beta \displaystyle\sum_{s'} h(s') \frac{\alpha}{p_T(x'_H, S'_H) C_T(m'_H, x'_H, S'_H)}$

where $\gamma = \int [\mu(x_G, S_G)m] \, d\lambda_G$ and $m'_H = m\mu(x_G)/\gamma$ and where we use the notation $p_N(x_G)$,$m(x_G)$, and $A(x_G)$ to recall that these variables are all recoverable from the state x_G of and the continuation histories m'_H, x'_H, and S'_H are induced by the sticky price firm decision rules $\bar{p}_N$ and the monetary policy rule $\bar{\mu}$.

More importantly, we next note that this problem can be split into a static part and a dynamic part. The static part is to solve

$$(114) \quad \max_{p_T, C_T(x_G), C_N(x_G)} \int \left[U \left(C_T(x_G), C_N(x_G), C_T(x_G) + \frac{C_N(x_G)}{A(x_G)} \right) \right] d\lambda_G$$

subject to (111) and (112).

For any given p_T, the dynamic part is to solve

$$(115) \quad \max_{\mu(x_G)} \sum_s h(s') \bar{W}(S'_G)$$

$$(116) \quad \gamma \frac{b}{p_T} = \beta \sum_{s'} h(s') \frac{\alpha}{p_T(x'_H, S'_H) C_T(m\mu(x_G)/\gamma, x'_H, S'_H)}$$

$$\gamma = \int [\mu(x_G, S_G) m] \, d\lambda_G$$

The reason we can separate these problems is that the value of the dynamic part is independent of p_T. To see why note that the aggregate growth rate of money is homogenous of degree 1 in $\mu(x_G)$ while the value $\bar{W}(S'_G)$ and the right hand side of the constraint (116) are homogenous of degree 0 in $\mu(x_G)$. Hence the value in (115) does not depend on p_T. Thus, for example, if $\tilde{p}_T$ is twice as large as p_T then the vector $\tilde{\mu}(x_G)$ and the associated aggregate growth rate $\tilde{\gamma}$ can both be chosen to be twice as $\mu(x_G)$ and γ and the values of the left side and the right side of (116) are unchanged. Intuitively, the value and allocations of a given country depend only on the ratio of money holdings of the country to the aggregate money holdings and this ratio, of course, does not depend on the absolute scale of money growth rates.

We prove a preliminary lemma that immediately implies Lemma 6.

Lemma A1. i) Under our utility function (8), if at the beginning of period $t \geq 1$ there is a non-degenerate money holding distribution then the date t cash-in-advance constraint has a zero multiplier for all m and all z and ii) If $\theta(s_1) > 1$ for all s_1 then in any continuation Markov equilibrium the multiplier on the cash-in-advance is binding for at least one level of aggregate shocks z and for a positive of measure of relative money holdings m in the support of λ_m. (CHECK NOTATION)

Proof of part i. Suppose by way of contradiction that the money holding distribution across countries is not degenerate in period t so that the beginning of period t money holdings in, say countries 1 and 2, satisfy $m_1 < m_2$ but the cash-in-advance constraint binds for country 1 for some realization of the shocks in period t. Note from (112) that the value of consumption of the traded good, $p_T C_{Ti} = \min[m_i, \alpha p_T/b]$ for $i = 1, 2$ does not vary with the idiosyncratic shock. It follows that $p_T C_{T1} \leq p_T C_{T1}$ with strict inequality for at least one aggregate state. It follows that

$$(117) \quad \sum_s h(s) \frac{1}{p_T(S_H) C_T(m_1, x_{H1}, S_H)} > \sum_s h(s) \frac{1}{p_T(S_H) C_T(m_2, x_{H2}, S_H)}$$

But the first order condition for money holdings from period $t-1$ to t implies that for both countries $i = 1, 2$

$$(118)\quad \frac{b}{\alpha p_T} = \beta \sum_s h(s) \frac{1}{p_T(S_H) C_T(m_i, x_{Hi}, S_H)}$$

where p_T is the price of traded goods in period $t-1$. Clearly, (117) contradicts (118). At an intuitive level, either the agent with the lower money holdings would want to hold more money or the agent with the higher money holdings would want to hold less money, but it cannot be that these agents want to hold differing amounts of money when there is a binding cash in advanced constraint in the next period.

Proof of part ii. Suppose by way of contradiction that the cash-in-advance constraint is slack for all countries for all realizations of the aggregate shock. Consider the static problem (114). The foc with respect to p_T evaluated the equilibrium rule (denoted by bars) implies that

$$(119)\quad 1 = \int \frac{\bar{p}_T(z, \lambda_G)}{A(x_G) \bar{p}_N(x_F, S_F)} d\lambda_G(x_G)$$

Now the sticky price first order condition evaluated in equilibrium is

$$(120)\quad \bar{p}_N(x_F, S_F) = \theta(s_1) \sum_{s_2} h^2(s_2) \frac{\bar{p}_T(z, \lambda_G)}{A(s_2)}$$

which since $\theta(s_1) > 1$ for all s_1 implies that

$$(121)\quad \sum_{s_2} h^2(s_2) \frac{\bar{p}_T(z, \lambda_G)}{A(s_2) \bar{p}_N(x_F, S_F)} < 1$$

Integrating (121) over the state x_F with respect to the measure λ_F implies that

$$(122)\quad \int \sum_{s_2} h^2(s_2) \frac{\bar{p}_T(z, \lambda_G)}{A(s_2) \bar{p}_N(x_F, S_F)} d\lambda_F(x_F) = \int \frac{\bar{p}_T(z, \lambda_G)}{A(x_G) p_N(x_G)} d\lambda_G(x_G) < 1$$

where in the first equality we have used the property that the marginal measure of λ_G over x_F is λ_F. The inequality in (122) contradicts (119). *Q.E.D.*

Note that the intuition for this lemma is similar to why the cash-in-advance constraint must be binding under flexible exchange rates. Suppose first there are no shocks and that the cash-in-advance constraint were slack. Then given some price for nontraded goods, the monetary authority could eliminate the monopoly distortion at no cost by raising the price of traded goods so that the product of the marginal rate of transformation between traded and nontraded goods and the inverse of their relative prices equals one. As condition (119) makes clear, with shocks it is optimal to make the average of this product equal to one. Now, in equilibrium the monopolist understands the the government will try eliminate this monopoly distortion and hence best responds with a higher price for which the wedge is not eliminated. Thus, such a situation cannot be an equilibrium. Rather in equilibrium the cash-in-advance constraint must bind for enough countries so that the benefits of raising the price of traded goods to correct the monopoly distortion just balance the costs of lowering the consumption of traded goods.

Combining parts i) and ii) of Lemma A1 immediately implies Lemma 2

Lemma 2. Under (8) if the markup is strictly positive in all states in that $\theta(s_1) > 1$ for all s_1 then in any Markov equilibrium with fixed exchange rates, given any initial distribution of money at the beginning of the period then the end of period money holdings are concentrated on a single point.

Proof: Suppose for contradiction that in a continuation Markov equilibrium the money holdings distribution, λ_m, is not degenerate. By part i) of Lemma A1, it must be that for all z and m in support of λ_m the multiplier on the cash-in-advance constraint is zero. This is a contradiction because by part ii) of Lemma A2 in any continuation Markov equilibrium the multiplier on the cash-in-advance is binding for at least one z and some m in the support of λ_m. *Q.E.D.*

D. Lemmas A2 and A3

We start with the characterization of the Markov equilibrium under flexible exchange rates.

Lemma A2. Under (8) the Markov equilibrium outcome with flexible exchange rates is such that the ratio $p_T(s^t)/m(s^{t-1})$ denoted $q_T(s_t)$ only depends on s_t and solves

(123)
$$q_T(s_t) = \max\left\{\frac{q_N(s_{1t})A(s_{2t})}{2(1-\alpha)}\left[(1-2\alpha)+\sqrt{(1-2\alpha)^2+4(1-\alpha)\frac{1}{A(s_{2t})}\frac{b}{q_N(s_{1t})}}\right],\frac{b}{\alpha}\right\},$$

the ratio $p_N(s^{t-1}, s_{1t})/m(s^{t-1})$ denoted $q_N(s_{1t})$ only depends on s_{1t} and solves

(124) $$q_N(s_{1t}) = \theta(s_{1t})\sum_{s_{2t}} h^2(s_{2t})\frac{q_T(s_t)}{A(s_{2t})},$$

furthermore, $C_T(s_t) = \min\left\{\frac{1}{q_T(s_t)}, \frac{\alpha}{b}\right\}$, and $C_N(s_t) = \frac{1-\alpha}{b}\frac{q_T(s_t)}{q_N(s_{1t})}$. Finally, the money growth rate is $\mu(s_t) = \frac{\beta\alpha}{b}q_T(s_t)$ and the inflation rate in sector $i = T, N$, defined as $\pi_i(z_{t-1}, z_t) = P_i(s_t)/P_i(s_{t-1})$, is $\pi_i(s_{t-1}, s_t) = \mu(s_{t-1})q_i(z_t)/q_i(z_{t-1})$.

Proof. Start by solving (38), which under (8), using (112) and (111) can be written as

$$\begin{aligned} W(S_G) \quad = \quad & \max_{p_T(x_G),\mu(x_G)}\int\left[\alpha\log\left(\min\left\{\frac{m}{p_T(x_G)},\frac{\alpha}{b}\right\}\right)+(1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{p_T(x_G)}{p_N}\right)\right]d\lambda_G \\ & -b\int\left[\min\left\{\frac{m}{p_T(x_G)},\frac{\alpha}{b}\right\}+\frac{1-\alpha}{bA(s_2)}\frac{p_T(x_G)}{p_N}\right]d\lambda_G+\beta\sum_s h(s')W\left(S'_G\right)\end{aligned}$$

subject to

$$\gamma\frac{b}{p_T(x_G)} = \beta\sum_{s'} h(s')\frac{\alpha}{p_T(x'_H, S'_H)C_T(m'_H, x'_H, S'_H)}$$

Now consider a change of variable: let

(125) $q_T(x_G) = p_T(x_G)/m \quad \text{and} \quad q_N = p_N/m$

and define S_G^q to be a measure over q_N. S_G^q is the relevant state variable for the problem, which can be rewritten as

$$\begin{aligned} W(S_G^q) \;=\; & \max_{q_T(x_G^q),\mu(x_G^q)} \int \left[\alpha \log\left(\min\left\{\frac{1}{q_T\left(x_G^q\right)}, \frac{\alpha}{b}\right\}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{q_T(x_G^q)}{q_N}\right)\right] d\lambda_G^q \\ & -b\int\left[\min\left\{\frac{1}{q_T(x_G^q)}, \frac{\alpha}{b}\right\} + \frac{1-\alpha}{bA(s_2)}\frac{q_T(x_G^q)}{q_N}\right] d\lambda_G^q + \beta\sum_s h(s')W\left(S_G^{q\prime}\right) \end{aligned}$$

subject to

$$\gamma\mu(x_G^q)\frac{b}{q_T(x_G^q)} = \beta\sum_{s'} h(s')\frac{\alpha}{q_T(x_H', S_H')C_T(m_H', x_H', S_H')}$$

Notice that the optimal $q_T(x_G^q)$ can be found by solving pointwise for all x_G^q is support of λ_G^q the following static problem: for all x_G^q

$$\begin{aligned} & \max_{q_T(x_G^q)} \left[\alpha \log\left(\min\left\{\frac{1}{q_T\left(x_G^q\right)}, \frac{\alpha}{b}\right\}\right) + (1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{q_T(x_G^q)}{q_N}\right)\right] \\ & -b\left[\min\left\{\frac{1}{q_T(x_G^q)}, \frac{\alpha}{b}\right\} + \frac{1-\alpha}{bA(s_2)}\frac{q_T(x_G^q)}{q_N}\right] \end{aligned}$$

or equivalently - dropping the dependence from x_G^q - and defining $x = q_T/q_N$ we can write

$$\max_x \left[\alpha\log\left(\frac{1}{xq_N}\right) + (1-\alpha)\log(x) - b\frac{1}{q_N x} - (1-\alpha)\frac{1}{A(s_2)}x\right]$$

$$= \max_x (1-2\alpha)\log(x) - \frac{b}{q_N x} - (1-\alpha)\frac{1}{A(s_2)}x + \text{constants}$$

subject to

(126) $x \geq \dfrac{b}{q_N\alpha}$

If (126) does not bind, the solution to the above problem satisfies:

$$0 = \frac{1-2\alpha}{x} - (1-\alpha)\frac{1}{A(s_2)} + \frac{b}{q_N}\frac{1}{x^2}$$

$$0 = (1-\alpha)\frac{1}{A(s_2)}x^2 - (1-2\alpha)\,x - \frac{b}{q_N}$$

Then the monetary authority best response is:

$$(127)\ \ x(q_N, s) = \max\left\{A(s_2)\frac{(1-2\alpha)+\sqrt{(1-2\alpha)^2+4(1-\alpha)\frac{1}{A(s_2)}\frac{b}{q_N}}}{2(1-\alpha)}, \frac{b}{q_N\alpha}\right\}$$

or

$$(128)\ \ q_T(q_N, s) = \max\left\{q_N A(s_2)\frac{(1-2\alpha)+\sqrt{(1-2\alpha)^2+4(1-\alpha)\frac{1}{A(s_2)}\frac{b}{q_N}}}{2(1-\alpha)}, \frac{b}{\alpha}\right\}$$

Now, from (**??**), in equilibrium it must be that the private best response to government $p_T^i(s)$:

$$(129)\ \ q_N(s_1) = \theta(s_1)\sum_{s_2} h^2(s_2)\frac{q_T(q_N(s_1), s)}{A(s_2)}$$

We can combine (128) and (129) to get

$$(130)\ \ 1 = \theta(s_1)\sum_{s_2} h^2(s_2)\max\left\{\frac{(1-2\alpha)+\sqrt{(1-2\alpha)^2+\frac{4}{A(s_2)}\frac{(1-\alpha)b}{q_N(s_1)}}}{2(1-\alpha)}, \frac{1}{A(s_2)q_N(s_1)}\frac{b}{\alpha}\right\}$$

or, if (126) never binds, simply

$$(131)\ \ 1 = \theta(s_1)\frac{(1-2\alpha)+\sum h^2(s_2)\sqrt{(1-2\alpha)^2+\frac{4}{A(s_2)}\frac{(1-\alpha)b}{q_N(s_1)}}}{2(1-\alpha)}$$

which implicitly defines $q_N(s_1)$. Using $q_N(s_1)$ in (128) gives and expression for the equilibrium $q_T(s)$ and finally the other relevant equilibrium objects can be recovered using $q_N(s_1)$ and $q_T(s)$ in (112) and (111). *Q.E.D.*

It turns out that it is particularly simple to characterize the Markov equilibrium with fixed exchange rates when the cash-in-advance constraint always holds with equality. It follows from the proof of Lemma 6 that a sufficient condition for this to be true is that there productivity shocks in the nontraded goods sector have no aggregate component, that is the set Z_2 is a singleton.

Lemma A3. Assume the all agents begin with the same initial holdings of money initial distribution of money, (8) holds, the markup is strictly positive in all states in that $\theta(s_1) > 1$ for all s_1, and the cash-in-advance constraint holds with equality in all states. Then the Markov equilibrium outcome in a monetary union is such that the prices of consumptions of nontraded and traded goods can be written as $p_N(s_{1t}), C_N(s_{1t}, z_{2t})$, $p_T(z_t)$, and $C_T(z_t)$ and solve

$$(132)\ \ p_N(s_{1t}) = \theta(s_{1t})\sum h^2(s_{2t})\frac{p_T(z_t)}{A(s_{2t})}$$

where

$$(133)\quad p_T(z_t) = \frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + 4(1-\alpha)b\sum_{\tilde{s}_t}\left[\frac{1}{A(\tilde{s}_{2t})p_N(\tilde{s}_{1t})}\right]h(\tilde{s}_t|z_t)}}{\sum_{\tilde{s}}\left[\frac{2(1-\alpha)}{A(\tilde{s}_2)p_N(\tilde{s}_1)}\right]h(\tilde{s}_t|z_t)}$$

furthermore $C_T(z_t) = 1/p_T(z_t)$ and $C_N(s_{1t}, z_{2t}) = \frac{1-\alpha}{b}\frac{p_T(z_t)}{p_N(s_{1t})}$. Finally, the aggregate money growth rate is $\gamma(z_t) = \frac{\beta\alpha}{b}p_T(z_t)$ and the inflation rate in sector $i = T, N$, defined as $\pi_i(z_{t-1}, z_t) = P_i(z_t)/P_i(z_{t-1})$, is $\pi_i(z_{t-1}, z_t) = \gamma(z_{t-1})p_i(z_t)/p_i(z_{t-1})$.

Proof. First if the distribution λ_G puts all mass on $m = 1$ and the cash-in-advance constraint holds with equality so that $p_T C_T = 1$ then problem (**??**) can be written for all (z, λ_G) as

$$(134)\quad \max_{p_T} \int \left[-\alpha\log(p_T) + (1-\alpha)\log\left(\frac{1-\alpha}{b}\frac{p_T}{p_N}\right) - b\frac{1}{p_T} - \frac{1-\alpha}{A(s_2)}\frac{p_T}{p_N}\right] d\lambda_G$$

$$= \max_{p_T}(1-2\alpha)\log(p_T) - b\frac{1}{p_T} - \int\left[\frac{1-\alpha}{A(s_2)}\frac{p_T}{p_N}\right] d\lambda_G + \text{constants}$$

where in (134) the integral is effectively over p_N and s_2. The solution to the problem above satisfies:

$$0 = \frac{1-2\alpha}{p_T} + b\left(\frac{1}{p_T}\right)^2 - \int \frac{(1-\alpha)}{A(s_2)}\frac{1}{p_N} d\lambda_G$$

or equivalently

$$0 = p_T^2\left[\int \frac{(1-\alpha)}{A(s_2)}\frac{1}{p_N} d\lambda_G\right] - (1-2\alpha)\,p_T - b$$

We can thus solve for the monetary authority best response to some given aggregate shock z and distribution λ_G is

$$(135)\quad \bar{p}_T(z, \lambda_G) = \frac{(1-2\alpha) + \sqrt{(1-2\alpha)^2 + 4b\left[\int \frac{(1-\alpha)}{A(s_2)}\frac{1}{p_N} d\lambda_G\right]}}{2\left[\int \frac{(1-\alpha)}{A(s_2)}\frac{1}{p_N} d\lambda_G\right]}$$

In equilibrium we must impose that (**??**) is satisfied. Substituting (135) into (**??**) for all s_1 and using that $\bar{p}_N(x_F, S_F) = \bar{p}_N(s_1)$ reduces (135) to (132). For all z_1,equations (132) for all ν_1 give rise to a system of equation in $\bar{p}_N(z_1, \nu_1)$ that can be solved, yielding the price of the non-traded good on the equilibrium path . Given the solution for $\bar{p}_N(s_1)$, $\bar{p}_T(z)$ can be determined from (135) as in (133). Finally, $C_T(s)$ and $C_N(s)$ can be recovered using (132) and (133) in (111), (112) with a cash-in-advance constraint holding with equality. *Q.E.D.*

E. Proof of Lemma 4

Lemma 4. Under (8) and constant productivity A^N, A^S, the allocations in the Markov equilibrium in a monetary union satisfy for $i = N, S$

$$C_T^i(s) = \frac{\alpha}{b} - \frac{1-\alpha}{b}\left(1 - \sum_{i=N,S}\lambda^i \sum_v g(v|z)\frac{1}{\theta^i(s_1)}\right) \tag{136}$$

and

$$C_N^i(s) = \frac{1-\alpha}{b}\frac{A^i}{\theta^i(z_{i1}, v)}$$

and $L^i(s) = C_T(s) + C_N(s)/A$.

Proof. The problem reduces to

$$\max_{p_T^i} \sum_{i=N,S}\lambda^i \sum_v g(v|z)\left[-\alpha \log p_T + (1-\alpha)\log \frac{1-\alpha}{b}\frac{p_T}{p_N^i(s_1)} - b\left(\frac{1}{p_T} + \frac{1-\alpha}{b}\frac{p_T}{p_N^i(s_1)}\frac{1}{A^i}\right)\right]$$

so we have that

$$0 = (1-2\alpha)\frac{1}{p_T} + b\frac{1}{p_T^2} - (1-\alpha)\sum_{i=N,S}\lambda^i \sum_v g(v|z)\frac{1}{p_N^i(s_1)A^i}$$

and from the sticky price foc we have

$$p_N^i(s_1) = \theta^i(s_1)\frac{p_T}{A^i}$$

so

$$0 = (1-2\alpha)\frac{1}{p_T} + b\frac{1}{p_T^2} - (1-\alpha)\sum_{i=N,S}\lambda^i \sum_v g(v|z)\frac{1}{\theta^i(s_1)\frac{p_T}{A^i}A^i}$$

$$0 = (1-2\alpha)C_T + bC_T^2 - (1-\alpha)C_T\sum_{i=N,S}\lambda^i \sum_v g(v|z)\frac{1}{\theta^i(s_1)}$$

$$C_T^i(s) = C_T(s) = \frac{\alpha}{b} - \frac{1-\alpha}{b}\left(1 - \sum_{i=N,S}\lambda^i \sum_v g(v|z)\frac{1}{\theta^i(s_1)}\right)$$